Now to define what we are actually creating with these two processes.

**Definition:** A *Basis* for the vector space $V$ is a linearly independent spanning set of $V$.

**Basic Examples:**

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\text{ for } \mathbb{R}^3
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\text{ for } M_{2 \times 2}
\]

and so on.

**Less Basic Examples:**

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\text{ for } \mathbb{R}^3
\]

\[
\{x^2 - 1, x^2 + x, x - 1\} \text{ for } \mathcal{P}_2.
\]

Now to look into the size of bases.

**Dimension**

**The Fundamental Theorem** (of vector spaces?)

Take a vector space

\[ V = \text{Span}\{v_1, v_2, v_3, \ldots, v_n\} \]

and the linearly independent set

\[ \{x_1, x_2, \ldots, x_k\} \in V. \]

(Notice that the $x$ go to $k$, the $v$ go to $n$.)

We always get $k \leq n$.

This is not particularly easy to prove, but hopefully will make some sense. This one is a proof by contradiction. The idea is you start with the *opposite* of what you want, then prove it to be impossible. So, we start by assuming that $k = n + 1$ then show that this makes the set of $x$ vectors linearly dependent.

**Proof.**

The first step is to write $x_1$ as a linear combination of the $v$ vectors:

\[ x_1 = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n. \]

This is possible since the $v$ vectors span all of $V$ and $x_1 \in V$. Also, $x_1 \neq 0$, since the set of them is not linearly dependent, so at least ONE of the $a_k$ values is $\neq 0$. So, rearrange the $v$ so that we get

\[ x_1 = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n, \quad a_1 \neq 0 \]