Properties:

• $A^{-1}$ exists implies $(A^T)^{-1} = (A^{-1})^T$.
• $A, B$, both invertible, implies $(AB)^{-1} = B^{-1}A^{-1}$.
• $A^{-1}$ exists means $(A^{-1})^{-1} = A$.
• $A$ invertible then $cA$ is too, for $c \in \mathbb{R} \neq 0$. $(cA)^{-1} = \frac{A^{-1}}{c}$.
• $I$ is invertible. $I^{-1} = I$.
• $A$ invertible then $A^k$ is invertible, $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ (use $A^0 = I$).
• If $A$ is invertible, its inverse is unique.

Now take a look at what this means for a linear system:

$$Ax = b, \quad A \text{ invertible} \implies A^{-1}Ax = A^{-1}b \implies x = A^{-1}b.$$ 

Having an invertible $A$ on the left hand side means you can get the solution that way (usually not worth the effort, just solve it), so

• You ALWAYS get a solution (always consistent).
• The solution is unique.

This is primarily of value when one has to calculate many solutions of the form $Ax = b$. Each can then be solved with a mere multiplication, nothing else.

Calculating Inverses: Not as hard as you may expect. First, it’s really easy for any matrix smaller than $3 \times 3$.

For a $1 \times 1$, $[a]^{-1} = \left[ \frac{1}{a} \right]$. Sorry, couldn’t resist.

For a $2 \times 2$: \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}.
\]

Example:

\[
\begin{bmatrix}
2 & -1 \\
4 & -3
\end{bmatrix}^{-1} = \frac{1}{-6+4} \begin{bmatrix}
-3 & 1 \\
-4 & 2
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} & -\frac{1}{2} \\
2 & -1
\end{bmatrix}.
\]

Once you’re dealing with a bigger matrix than that, you have to row reduce. For $A$, an $n \times n$ matrix, simply solve the linear system $[A|I]$, row reduce the left down to $I$ and the right will be $A^{-1}$. 

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