**Remark** A similar proof may be given for the continuity of cosine function.

**Example 18** Prove that the function defined by \( f(x) = \tan x \) is a continuous function.

**Solution** The function \( f(x) = \tan x = \frac{\sin x}{\cos x} \). This is defined for all real numbers such that \( \cos x \neq 0 \), i.e., \( x \neq (2n + 1) \frac{\pi}{2} \). We have just proved that both sine and cosine functions are continuous. Thus \( \tan x \) being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if \( f \) and \( g \) are two real functions, then

\[
(f \circ g)(x) = f(g(x))
\]

is defined whenever the range of \( g \) is a subset of domain of \( f \). The following theorem (stated without proof) captures the continuity of composite functions.

**Theorem 2** Suppose \( f \) and \( g \) are real valued functions such that \( (f \circ g) \) is defined at \( c \). If \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \( (f \circ g) \) is continuous at \( c \).

The following examples illustrate this theorem.

**Example 19** Show that the function defined by \( f(x) = \sin(x^2) \) is a continuous function.

**Solution** Observe that the function is defined for every real number. The function \( f \) may be thought of as a composition \( g \circ h \) of the two functions \( g \) and \( h \), where \( g(x) = \sin x \) and \( h(x) = x^2 \). Since \( g \) and \( h \) are continuous functions, by Theorem 2, it can be deduced that \( f \) is a continuous function.

**Example 20** Show that the function \( f \) defined by

\[
f(x) = |1 - x| + |x|,
\]

where \( x \) is any real number, is a continuous function.

**Solution** Define \( g \) by \( g(x) = 1 - x + |x| \) and \( h \) by \( h(x) = |x| \) for all real \( x \). Then

\[
(h \circ g)(x) = h(g(x))
\]

\[
= h(1 - x + |x|)
\]

\[
= |1 - x + |x|| = f(x)
\]

In Example 7, we have seen that \( h \) is a continuous function. Hence \( g \) being a sum of a polynomial function and the modulus function is continuous. But then \( f \) being a composite of two continuous functions is continuous.
or \[ \frac{dy}{dx} = y \log a \]

Thus \[ \frac{d}{dx}(a^x) = a^x \log a \]

Alternatively \[ \frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \log a}) = e^{x \log a} \frac{d}{dx}(x \log a) \]

\[ = e^{x \log a} \cdot \log a = a^x \log a. \]

**Example 32** Differentiate \( x^{\sin x}, x > 0 \) w.r.t. \( x \).

**Solution** Let \( y = x^{\sin x} \). Taking logarithm on both sides, we have

\[ \log y = \sin x \log x \]

Therefore

\[ \frac{1}{y} \frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x) \]

or

\[ \frac{1}{y} \frac{dy}{dx} = (\sin x) \frac{1}{x} - \cos x \log x \]

or

\[ \frac{dy}{dx} = y \left( \frac{\sin x}{x} + \cos x \log x \right) \]

\[ = x^{\sin x - 1} \cdot \sin x + x^{\sin x} \cdot \cos x \log x \]

**Example 33** Find \( \frac{dy}{dx} \), if \( y^a + x^b + x^c = a^b \).

**Solution** Given that \( y^a + x^b + x^c = a^b \).

Putting \( u = y^a, v = x^b \) and \( w = x^c \), we get \( u + v + w = a^b \)

Therefore \[ \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \] ... (1)

Now, \( u = y^a \). Taking logarithm on both sides, we have

\[ \log u = x \log y \]

Differentiating both sides w.r.t. \( x \), we have
\[ \frac{1}{u} \frac{du}{dx} = x \frac{d}{dx} (\log y) + \log y \frac{d}{dx} (x) \]
\[ = x \frac{1}{y} \frac{dy}{dx} + \log y \cdot 1 \]

So
\[ \frac{du}{dx} = u \left( \frac{x}{y} \frac{dy}{dx} + \log y \right) = x^y \left[ \frac{x}{y} \frac{dy}{dx} + \log y \right] \quad \cdots (2) \]

Also \( v = x^y \)

Taking logarithm on both sides, we have
\[ \log v = y \log x \]

Differentiating both sides w.r.t. \( x \), we have
\[ \frac{1}{v} \frac{dv}{dx} = y \frac{d}{dx} (\log x) + \log x \frac{dy}{dx} \]
\[ = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} \]

So
\[ \frac{dv}{dx} = x^y \left[ \frac{1}{x} + \log x \frac{dy}{dx} \right] \quad \cdots (3) \]

Again \( w = x^x \)

Taking logarithm on both sides, we have
\[ \log w = x \log x \]

Differentiating both sides w.r.t. \( x \), we have
\[ \frac{1}{w} \frac{dw}{dx} = x \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (x) \]
\[ = x \cdot \frac{1}{x} + \log x \cdot 1 \]

\[ \frac{dw}{dx} = w (1 + \log x) \]
\[ = x^x (1 + \log x) \quad \cdots (4) \]
From (1), (2), (3), (4), we have
\[ y^x \left( \frac{x \, dy}{y \, dx} + \log y \right) + x^y \left( \frac{y}{x} + \log x \frac{dy}{dx} \right) + x^y (1 + \log x) = 0 \]
or
\[ (x \cdot y^{x-1} + x \cdot \log x) \frac{dy}{dx} = -(x^y (1 + \log x) - x^y - y^x \log y) \]
Therefore
\[ \frac{dy}{dx} = \frac{-[y^x \log y + y \cdot y^{x-1} + x^y (1 + \log x)]}{x \cdot y^{x-1} + x^y \log x} \]

**EXERCISE 5.5**

Differentiate the functions given in Exercises 1 to 11 w.r.t. \( x \).

1. \( \cos x \cdot \cos 2x \cdot \cos 3x \)
2. \( \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \)
3. \( (\log x)^{\sin x} \)
4. \( x^x = 2^{\sin x} \)
5. \( (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4 \)
6. \( (\log x)^x + x^{\log x} \)
7. \( (\sin x)^x + \sin^x \)
8. \( (\sin x)^x + \sin^x \)
9. \( (x \cos x)^y + (x \sin x)^x \)
10. \( x^{\cos x} + x^2 + 1 \) \( \frac{1}{x^2 - 1} \)

Find \( \frac{dy}{dx} \) of the functions given in Exercises 12 to 15.

12. \( x^y + y^x = 1 \)
13. \( y^x = x^y \)
14. \( (\cos x)^y = (\cos y)^x \)
15. \( xy = e^{x-y} \)

Find the derivative of the function given by \( f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \) and hence find \( f'(1) \).

17. Differentiate \( (x^2 - 5x + 8)(x^3 + 7x + 9) \) in three ways mentioned below:
   (i) by using product rule
   (ii) by expanding the product to obtain a single polynomial.
   (iii) by logarithmic differentiation.

Do they all give the same answer?
Therefore
\[ \frac{dy}{dx} = \frac{dy}{d\theta} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\tan \theta = -\sqrt{\frac{y}{x}} \]

**Note** Had we proceeded in implicit way, it would have been quite tedious.

**EXERCISE 5.6**

If \( x \) and \( y \) are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find \( \frac{dy}{dx} \).

1. \( x = 2a t^2, \quad y = at^4 \)
2. \( x = a \cos \theta, \quad y = b \cos \theta \)
3. \( x = \sin t, \quad y = \cos 2t \)
4. \( x = 4t, \quad y = \frac{4}{t} \)
5. \( x = \cos \theta - \cos 2\theta, \quad y = \sin \theta - \sin 2\theta \)
6. \( x = a (\theta - \sin \theta), \quad y = a (1 + \cos \theta) = \frac{\cos \theta}{\sqrt{\cos 2t}}, \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}} \)
7. \( x = a (\cos \theta + \theta \sin \theta), \quad y = b (\sin \theta - \theta \cos \theta) \)
8. \( x = \sqrt{a \sin^2 t}, \quad y = \sqrt{a \cos^2 t}, \quad \text{show that} \quad \frac{dy}{dx} = -\frac{y}{x} \)

**5.7 Second Order Derivative**

Let \( y = f(x) \). Then
\[ \frac{dy}{dx} = f'(x) \]

If \( f'(x) \) is differentiable, we may differentiate (1) again w.r.t. \( x \). Then, the left hand side becomes \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \) which is called the second order derivative of \( y \) w.r.t. \( x \) and is denoted by \( \frac{d^2 y}{dx^2} \). The second order derivative of \( f(x) \) is denoted by \( f''(x) \). It is also
we need to find all $x$ such that \( \frac{2^{x+1}}{1+4^x} \leq 1 \), i.e., all $x$ such that $2^{x+1} \leq 1 + 4^x$. We may rewrite this as $2 \leq \frac{1}{2^x} + 2^x$ which is true for all $x$. Hence the function is defined at every real number. By putting $2^x = \tan \theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[ \frac{2^{x+1}}{1+4^x} \right]$$

$$= \sin^{-1}\left[ \frac{2 \cdot 2}{1+(2^x)^2} \right]$$

$$= \sin^{-1}\left[ \frac{2 \tan \theta}{1+\tan^2 \theta} \right]$$

$$= \sin^{-1} [\sin 2\theta]$$

$$= 2\theta = 2 \tan^{-1} (2^x)$$

Thus

$$f'(x) = 2 \cdot \frac{\frac{d}{dx} (2^x)}{1+(2^x)^2} \cdot \frac{2}{1+4^x} \cdot (2^x) \log 2$$

$$= \frac{2^{x+1} \log 2}{1+4^x}$$

**Example 46** Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

**Solution** The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log (\sin x)^{\sin x} = \sin x \log (\sin x)$$

Then

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} (\sin x \log (\sin x))$$

$$= \cos x \log (\sin x) + \sin x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x)$$

$$= \cos x \log (\sin x) + \cos x$$

$$= (1 + \log (\sin x)) \cos x$$
Chain rule is a rule to differentiate composites of functions. If $f = v \circ u, t = u(x)$, and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then

$$ \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} $$

Following are some of the standard derivatives (in appropriate domains):

$$ \frac{d}{dx} \left( \sin^{-1} x \right) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \left( \cos^{-1} x \right) = -\frac{1}{\sqrt{1-x^2}} $$

$$ \frac{d}{dx} \left( \tan^{-1} x \right) = \frac{1}{1+x^2} \quad \frac{d}{dx} \left( \cot^{-1} x \right) = -\frac{1}{1+x^2} $$

$$ \frac{d}{dx} \left( \sec^{-1} x \right) = \frac{1}{x \sqrt{1-x^2}} \quad \frac{d}{dx} \left( \cosec^{-1} x \right) = -\frac{1}{x \sqrt{1-x^2}} $$

$$ \frac{d}{dx} (e^x) = e^x \quad \frac{d}{dx} (k x^a) = k x^{a-1} $$

Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = u(x)^v$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.

**Rolle’s Theorem**: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$ such that $f(a) = f(b)$, then there exists some $c$ in $(a, b)$ such that $f'(c) = 0$.

**Mean Value Theorem**: If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists some $c$ in $(a, b)$ such that

$$ f'(c) = \frac{f(b) - f(a)}{b - a} $$