8.4
Integration of Rational Functions by Partial Fractions

Method of Partial Fractions (f(x)/g(x) Proper)
1. Let \( x = r \) be a linear factor of \( g(x) \). Suppose that \( (x - r)^n \) is the highest power of \( x - r \) that divides \( g(x) \). Then, to this factor, assign the sum of the \( n \) partial fractions:

\[
\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_n}{(x - r)^n}
\]

Do this for each distinct linear factor of \( g(x) \).

2. Let \( x^2 + px + q \) be an irreducible quadratic factor of \( g(x) \) so that \( x^3 + px^2 + qx \) has no real roots. Suppose that \( (x^2 + px + q)^n \) is the highest power of this factor that divides \( g(x) \). Then, to this factor, assign the sum of the \( n \) partial fractions:

\[
\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}
\]

Do this for each distinct quadratic factor of \( g(x) \).

3. Set the original fraction \( f(x)/g(x) \) equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of \( x \).

4. Equate the coefficients of corresponding powers of \( x \) and solve the resulting equations for the undetermined coefficients.

Heaviside Method
1. Write the quotient with \( g(x) \) factored:

\[
\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}
\]

2. Cover the factors \( (x - r_i) \) of \( g(x) \) one at a time, each time replacing all the uncovered \( x \)'s by the number \( r_i \). This gives a number \( A_i \) for each root \( r_i \):

\[
A_1 = \frac{f(r_1)}{(r_1 - r_2)(r_1 - r_3)\cdots(r_1 - r_n)}
\]

\[
A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}
\]

\[
\vdots
\]

\[
A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}
\]

3. Write the partial fraction expansion of \( f(x)/g(x) \) as:

\[
\frac{f(x)}{g(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}
\]