Contents

31. To avoid the introduction of fractions we interchange rows one and three. The remaining operations are used to change all pivots to ones and eliminate nonzero entries above and below them.

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & -1 & 2 \\ 3 & -1 & 1 \\ 3 & 3 & 1 \end{bmatrix} \xrightarrow{3R_1 + R_2 \rightarrow R_2} \begin{bmatrix} -1 & -1 & 2 \\ 0 & -4 & 6 \\ 3 & 3 & 1 \end{bmatrix} \xrightarrow{3R_1 + R_3 \rightarrow R_3}$$

$$\begin{bmatrix} -1 & -1 & 2 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow{\frac{1}{7}R_3 \rightarrow R_3} \begin{bmatrix} -1 & -1 & 2 \\ 0 & -4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-6)R_3 + R_2 \rightarrow R_2} \begin{bmatrix} -1 & -1 & 2 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3 + R_1 \rightarrow R_1}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2 \rightarrow R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1 \rightarrow R_1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_1 \rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. The matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & -3 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

33. The matrix in reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

34. The matrix in reduced row echelon form is $\begin{bmatrix} 1 & 0 & \frac{3}{8} \\ 0 & 1 & -\frac{1}{4} \end{bmatrix}$.

35. The matrix in reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

36. The matrix in reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{6}{5} \\ 0 & 0 & 1 & \frac{8}{5} \end{bmatrix}$.

37. The augmented matrix for the linear system and the reduced row echelon form are

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right].$$

The unique solution to the linear system is $x = -1, y = 2$.

38. The augmented matrix for the linear system

$$\left[\begin{array}{cc|c} -3 & 1 & 1 \\ 4 & 2 & 0 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{2}{5} \end{array}\right].$$

The unique solution to the linear system is $x = -\frac{1}{5}, y = \frac{2}{5}$.

39. The augmented matrix for the linear system and the reduced row echelon form are

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 3 \\ 4 & -1 & -3 & 3 \\ -2 & -2 & 0 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right].$$

The unique solution for the linear system is $x = 1, y = 0, z = \frac{1}{3}$.

40. The augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 0 & -4 & 1 \\ 4 & 3 & -2 & 0 \\ 2 & 0 & 2 & 2 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{6} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right].$$

The unique solution for the linear system is $x = \frac{5}{6}, y = -1, z = \frac{1}{6}$.

41. The augmented matrix for the linear system and the reduced row echelon form are

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

will equal $\begin{bmatrix} -5 & 6 \\ 12 & 16 \end{bmatrix}$ if and only $b+2=6, 3a=12$, and $ab=16$. That is, $a=b=4$.

32. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Since

$$AB - BA = \begin{bmatrix} bg - cf & (af+bh-(be+fd) \\ (ce+dg)-(ag+ch) & cf - bg \end{bmatrix},$$

then the sum of the terms on the diagonal is $(bg - cf) + (cf - bg) = 0$.

33. Several powers of the matrix A are given by

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that if n is even, then A^n is the identity matrix, so in particular $A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Notice also that, if n is odd, then $A^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

34. Since $(A+B)(A-B) = A^2 - AB + BA - B^2$, then $(A+B)(A-B) = A^2 - B^2$ when $AB = BA$.

35. We can first rewrite the expression A^2B as $A^2B = AAB$. Since $AB = BA$, then $A^2B = AAB = ABA = BAA = BA^2$.

36. **a.** Since $AB = BA$ and $AC = CA$, then $(BC)A = B(CA) = B(AC) = A(BC)$ and hence BC and A commute. **b.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that A commutes with every 2×2 matrix. Then select any two matrices that do not commute. For example, let $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

37. Multiplication of A times the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ gives the first column vector of the matrix A. Then $A\mathbf{x} = \mathbf{0}$ forces the first column vector of A to be the zero vector. Then let $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ and so on, to show that each column vector of A is the zero vector. Hence, A is the zero matrix.

38. Let $A_n = \begin{bmatrix} 1-n & -n \\ n & 1+n \end{bmatrix}$ and $A_m = \begin{bmatrix} 1-m & -m \\ m & 1+m \end{bmatrix}$. Then

$$A_nA_m = \begin{bmatrix} (1-n)(1-m) - nm & (1-n)(-m) - (1+m)n \\ n(1-m) + m(1+n) & -mn + (1+n)(1+m) \end{bmatrix} = \begin{bmatrix} 1-(m+n) & -(m+n) \\ m+n & 1+(m+n) \end{bmatrix} = A_{m+n}.$$

39. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then

$$AA^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if and only if $a^2+b^2=0, c^2+d^2=0$, and $ac+bd=0$. The only solution to these equations is $a=b=c=d=0$, so the only matrix that satisfies $AA^t = \mathbf{0}$ is the 2×2 zero matrix.

7. Since the matrix A is row equivalent to the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the matrix A can not be reduced to the identity and hence is not invertible.

8. Since the matrix A is not row equivalent to the identity matrix, then A is not invertible.

9. $A^{-1} = \begin{bmatrix} 1/3 & -1 & -2 & 1/2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$

10. $A^{-1} = \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

11. $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

12.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1/2 & -1/2 & -1/2 & 0 \\ 1 & 1 & 0 & 1/2 \end{bmatrix}$$

13. The matrix A is not invertible.

14. The matrix A is not invertible.

15. $A^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$

16. The matrix A is not invertible.

17. Performing the operations, we have that $AB+A = \begin{bmatrix} 3 & 8 \\ 10 & -10 \end{bmatrix} = A(B+I)$ and $AB+B = \begin{bmatrix} 2 & 9 \\ 6 & -3 \end{bmatrix} = (A+I)B$.

18. Since the distributive property holds for matrix multiplication and addition, we have that $(A+I)(A+I) = A^2 + A + A + I = A^2 + 2A + I$.

19. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. **a.** Since $A^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$ and $-2A = \begin{bmatrix} -2 & -4 \\ 4 & -2 \end{bmatrix}$, then $A^2 - 2A + 5I = 0$. **b.** Since $(1)(1) - (2)(-2) = 5$, the inverse exists and $A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}(2I - A)$.
c. If $A^2 - 2A + 5I = 0$, then $A^2 - 2A = -5I$, so that $A\left(\frac{1}{5}(2I - A)\right) = \frac{2}{5}A - \frac{1}{5}A^2 = -\frac{1}{5}(A^2 - 2A) = -\frac{1}{5}(-5I) = I$. Hence $A^{-1} = \frac{1}{5}(2I - A)$.

20. Applying the operations $(-3)R_1 + R_2 \to R_2$ and $(-1)R_1 + R_3 \to R_3$ gives $\begin{bmatrix} 1 & \lambda & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 2-3\lambda & 0 \\ 1 & 2 & 1 \end{bmatrix}$. So if $\lambda = \frac{2}{3}$, then the matrix can not be reduced to the identity and hence, will not be invertible.

21. The matrix is row equivalent to $\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1-2\lambda & 1 \end{bmatrix}$. If $\lambda = -2$, then the second and third rows are identical, so the matrix can not be row reduced to the identity and hence, is not invertible.

22. Since the matrix is row equivalent to $\begin{bmatrix} 1 & 2 & 1 \\ 0 & \lambda-4 & -1 \\ 0 & 4-2\lambda & 0 \end{bmatrix}$, if $\lambda = 2$, then the matrix can not be row reduced to the identity matrix and hence, is not invertible.

23. **a.** If $\lambda \neq 1$, then the matrix A is invertible.

b. When $\lambda \neq 1$ the inverse matrix is $A^{-1} = \begin{bmatrix} -\frac{1}{\lambda-1} & \frac{\lambda}{\lambda-1} & -\frac{\lambda}{\lambda-1} \\ \frac{1}{\lambda-1} & -\frac{1}{\lambda-1} & \frac{1}{\lambda-1} \\ 0 & 0 & 1 \end{bmatrix}$.

27. Using the LU factorization $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, we have that

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

28.

$$A^{-1} = (LU)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -1 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

29. Suppose

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives the system of equations $ad = 0, ae = 1, bd = 1, be + cf = 0$. The first two equations are satisfied only when $a \neq 0$ and $d = 0$. But this incompatible with the third equation.

30. Since A is row equivalent to B there are elementary matrices such that $B = E_m \ldots E_1 A$ and since B is row equivalent to C there are elementary matrices such that $C = D_n \ldots D_1 B$. Then $C = D_n \ldots D_1 B = D_n \ldots D_1 E_m \ldots E_1 A$ and hence, A is row equivalent to C.

31. If A is invertible, there are elementary matrices $E_1, \ldots, E_k$ such that $I = E_k \cdots E_1 A$. Similarly, there are elementary matrices $D_1, \ldots, D_\ell$ such that $I = D_\ell \cdots D_1 B$. Then $A = E_k^{-1} \cdots E_1^{-1} D_\ell \cdots D_1 B$, so A is row equivalent to B.

32. a. Since L is invertible, the diagonal entries are all nonzero. **b.** The determinant of A is the product of the diagonal entries of L and U, that is $\det(A) = \ell_{11} \cdots \ell_{nn} u_{11} \cdots u_{nn}$. **c.** Since L is lower triangular and invertible it is row equivalent to the identity matrix and can be reduced to I using only replacement operations.

Exercise Set 1.8

1. We need to find positive whole numbers x_1, x_2, x_3, and x_4 such that $x_1 Al_3 + x_2 CuO \longrightarrow x_3 Al_2O_3 + x_4 Cu$ is balanced. That is, we need to solve the linear system

$$\begin{cases} 3x_1 &= 2x_3 \\ x_2 &= 3x_3 \\ x_2 &= x_4 \end{cases}, \text{ which has infinitely many solutions given by } x_1 = \frac{2}{9}x_2, x_3 = \frac{1}{3}x_2, x_4 = x_2, x_2 \in \mathbb{R}.$$

A particular solution that balances the equation is given by $x_1 = 2, x_2 = 9, x_3 = 3, x_4 = 9$.

2. To balance the equation $x_1 I_2 + x_2 Na_2S_2O_3 \longrightarrow x_3 NaI + x_4 Na_2S_4O_6$, we solve the linear system $\begin{cases} 2x_1 &= x_3 \\ 2x_2 &= x_3 + 2x_4 \\ 2x_2 &= 4x_4 \\ 3x_2 &= 6x_4 \end{cases}$, so that $x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4, x_4 \in \mathbb{R}$. For a particular solution that balances the equation, let $x_4 = 1$, so $x_1 = 1, x_2 = 2$, and $x_3 = 2$.

16. T

17. T

18. F. Since the column vectors are linearly independent, $\det(A) \neq 0$

19. T

20. T

21. F. If the vector $\mathbf{v_3}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, then the vectors will be linearly dependent.

22. F. At least one is a linear combination of the others.

23. F. The determinant of the matrix will be zero since the column vectors are linearly dependent.

24. F. The third vector is a combination of the other two and hence, the three together are linearly dependent.

25. T

26. F. An $n \times n$ matrix is invertible if and only if the column vectors are linearly independent.

27. T

28. F. For example, the column vectors of any 3×4 matrix are linearly dependent.

29. T

30. F. The vector can be a linear combination of the linearly independent vectors $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$.

31. T

32. F. The set of coordinate vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is linearly independent, but the set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}\}$ is linearly dependent.

33. T

- Two linearly dependent vectors can not span $\mathbb{R}^2$. Let $S = \textbf{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}\right\}$. If $\mathbf{v}$ in in S, then there are scalars c_1 and c_2 such that
$$\mathbf{v} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left(-2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = (c_1 - 2c_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
and hence, every vector in the span of S is a linear combination of only one vector.

- A linearly dependent set of vectors can span a vector space. For example, let $S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$. Since the coordinate vectors are in S, then $\textbf{span}(S) = \mathbb{R}^2$, but the vectors are linearly dependent since $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In general, to determine whether or not a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is in $\textbf{span}\{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$, start with the vector equation
$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k} = \mathbf{v},$$
and then solve the resulting linear system. These ideas apply to all vector spaces not just the Euclidean spaces. For example, if $S = \{A \in M_{2\times 2} | \, A \text{ is invertible}\}$, then S is not a subspace of the vector space of all 2×2 matrices. For example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are both invertible, so are in S, but $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible. To determine whether of not $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ is in the span of the two matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, start with the equation
$$c_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 - c_2 & 2c_1 \\ c_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$
The resulting linear system is $c_1 - c_2 = 2, 2c_1 = -1, c_2 = 1, c_1 + c_2 = 1$, is inconsistent and hence, the matrix is not in the span of the other two matrices.

Solutions to Exercises

1. Let $\begin{bmatrix} 0 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y_2 \end{bmatrix}$ be two vectors in S and c a scalar. Then $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} + c \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_1 + cy_2 \end{bmatrix}$ is in S, so S is a subspace of $\mathbb{R}^2$.

2. The set S is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \notin S$.

3. The set S is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin S$.

4. The set S is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \notin S$ since $(3/2)^2 + 0^2 = 9/4 > 1$.

5. The set S is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $c = 0$, then $c\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$.

6. The set S is a subspace since $\begin{bmatrix} x \\ 3x \end{bmatrix} + c \begin{bmatrix} y \\ 3y \end{bmatrix} = \begin{bmatrix} x + cy \\ 3(x + cy) \end{bmatrix} \in S$.

$B = \left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \right\}$. Observe that $\mathbf{span}(S) = \mathbb{R}^3$.

30. The vectors can not be a basis since a set of four vectors in $\mathbb{R}^3$ is linearly dependent. To trim the set down to a basis for the span row reduce the matrix with column vectors the vectors in S. This gives

$$\begin{bmatrix} 2 & 1 & 0 & 2 \\ 2 & -1 & 2 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

A basis for the span consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for the span of S is given by

$B = \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$. Observe that $\mathbf{span}(S) = \mathbb{R}^3$.

31. Form the 3×5 matrix with first two column vectors the vectors in S and then augment the identity matrix. Reducing this matrix, we have that

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 \end{bmatrix}.$$

A basis for $\mathbb{R}^3$ consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for $\mathbb{R}^3$ containing S is $B = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

32. Form the 3×5 matrix with first two column vectors the vectors in S and then augment the identity matrix. Reducing this matrix, we have that

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

A basis for $\mathbb{R}^3$ consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for $\mathbb{R}^3$ containing S is $B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

33. A basis for $\mathbb{R}^4$ containing S is

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

34. A basis for $\mathbb{R}^4$ containing S is

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

35. A basis for $\mathbb{R}^3$ containing S is

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

36. A basis for $\mathbb{R}^3$ containing S is

$$B = \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

37. Let $\mathbf{e_{ii}}$ denote the $n \times n$ matrix with a 1 in the row i, column i component and 0 in all other locations. Then $B = \{\mathbf{e_{ii}} \mid 1 \le i \le n\}$ is a basis for the subspace of all $n \times n$ diagonal matrices.

19. Since the equation $c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ gives

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -a-b+c \\ 0 & 1 & 0 & a+b \\ 0 & 0 & 1 & a+2b-c \end{array}\right], \text{ we have that } \begin{bmatrix} a \\ b \\ c \end{bmatrix}_B = \begin{bmatrix} -a-b+c \\ a+b \\ a+2b-c \end{bmatrix}.$$

20. Since

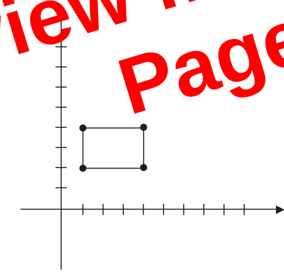
$$\begin{bmatrix} 1 & 0 & 0 & -1 & a \\ 0 & -1 & -1 & 0 & b \\ 1 & 1 & -1 & 0 & c \\ 0 & -1 & 0 & -1 & d \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & 0 & 0 & 2a+b-c-2d \\ 0 & 1 & 0 & 0 & -a-b+c+d \\ 0 & 0 & 1 & 0 & a-c-d \\ 0 & 0 & 0 & 1 & a+b-c-2d \end{bmatrix}, \text{ then } \left[\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right]_B = \begin{bmatrix} 2a+b-c-2d \\ -a-b+c+d \\ a-c-d \\ a+b-c-2d \end{bmatrix}.$$

21. **a.** $[I]_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

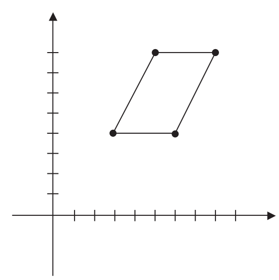
22. **a.** $[I]_{B_1}^{B_2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ **b.** $[I]_{B_2}^{B_1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ **c.** Since $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $([I]_{B_1}^{B_2})^{-1} = [I]_{B_2}^{B_1}$.

23. **a.** $[I]_S^B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 4 \end{bmatrix}_B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$; $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$; $\begin{bmatrix} 4 \\ 4 \end{bmatrix}_B = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

c.



d.



24. **a.** $[\mathbf{v}]_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$

- If A is an $m \times n$ matrix, then $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- $T(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) + \cdots + c_nT(\mathbf{v_n})$

The third property can be used to find the image of a vector when the action of a linear transformation is known only on a specific set of vectors, for example on the vectors of a basis. For example, suppose that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation and

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\0\end{bmatrix}, T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\-1\end{bmatrix}.$$

Then the image of an arbitrary input vector can be found since $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$. For example, let's find the image of the vector $\begin{bmatrix}1\\-2\\0\end{bmatrix}$. The first step is to write the input vector in terms of the basis vectors, so

$$\begin{bmatrix}1\\-2\\0\end{bmatrix} = -\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Then use the linearity properties of T to obtain

$$\begin{aligned} T\left(\begin{bmatrix}1\\-2\\0\end{bmatrix}\right) &= T\left(-\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + 2T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) \\ &= -\begin{bmatrix}-1\\2\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\-3\\3\end{bmatrix}. \end{aligned}$$

Solutions to Exercises

1. Let $\mathbf{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ be vectors in $\mathbb{R}^2$ and c a scalar. Since

$$T(\mathbf{u} + c\mathbf{v}) = T\left(\begin{bmatrix}u_1 + cv_1\\u_2 + cv_2\end{bmatrix}\right) = \begin{bmatrix}u_2 + cv_2\\u_1 + cv_1\end{bmatrix} = \begin{bmatrix}u_2\\u_1\end{bmatrix} + c\begin{bmatrix}v_2\\v_1\end{bmatrix} = T(\mathbf{u}) + cT(\mathbf{v}),$$

then T is a linear transformation.

2. Let $\mathbf{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ be vectors in $\mathbb{R}^2$ and c a scalar. Then

$$T(\mathbf{u} + c\mathbf{v}) = T\left(\begin{bmatrix}u_1 + cv_1\\u_2 + cv_2\end{bmatrix}\right) = \begin{bmatrix}(u_1 + cv_1) + (u_2 + cv_2)\\(u_1 + cv_1) - (u_2 + cv_2) + 2\end{bmatrix}$$

and

$$T(\mathbf{u}) + cT(\mathbf{v}) = \begin{bmatrix}(u_1 + cv_1) + (u_2 + cv_2)\\(u_1 + cv_1) - (u_2 + cv_2) + 4\end{bmatrix}.$$

For example, if $\mathbf{u} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}0\\1\end{bmatrix}$, and $c = 1$, then $T(\mathbf{u} + \mathbf{v}) = \begin{bmatrix}2\\2\end{bmatrix}$ and $T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix}2\\4\end{bmatrix}$. Hence, T is not a linear transformation.

is a linear transformation. The null space of T, denoted by $N(T)$, is the null space of the matrix, $N(A) = \{\mathbf{x} \in \mathbb{R}^3 \mid A\mathbf{x} = \mathbf{0}\}$. Since

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix},$$

the range of T, denoted by $R(T)$ is the column space of A, $\mathbf{col}(A)$. Since

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions given by $x_1 = -\frac{3}{2}x_3, x_2 = \frac{1}{2}x_3$, and x_3 a free variable. So the null space is $\left\{ t\begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$, which is a line that passes through the origin in three space. Also since the pivots in the reduced matrix are in columns one and two, a basis for the range is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ and hence, the range is a plane in three space. Notice that in this example, $3 = \dim(\mathbb{R}^3) = \dim(R(T)) + \dim(N(T))$. This is a fundamental theorem that if $T : V \longrightarrow W$ is a linear transformation defined on finite dimensional vector spaces, then

$$\dim(V) = \dim(R(T)) + \dim(N(T)).$$

If the mapping is given as a matrix product $T(\mathbf{v}) = A\mathbf{v}$ such that A is a $m \times n$ matrix, then this result is written as

$$n = \mathbf{rank}(A) + \mathbf{nullity}(A).$$

A number of useful statements are added to the list of equivalences concerning $n \times n$ linear systems:

A is invertible $\Leftrightarrow A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \Leftrightarrow A\mathbf{x} = \mathbf{0}$ has only the trivial solution
$\Leftrightarrow A$ is row equivalent to $I \Leftrightarrow \det(A) \neq 0 \Leftrightarrow$ the column vectors of A are linearly independent
$\Leftrightarrow$ the column vectors of A span $\mathbb{R}^n \Leftrightarrow$ the column vectors of A are a basis for $\mathbb{R}^n$
$\Leftrightarrow \mathbf{rank}(A) = n \Leftrightarrow R(A) = \mathbf{col}(A) = \mathbb{R}^n \Leftrightarrow N(A) = \{\mathbf{0}\} \Leftrightarrow \mathbf{row}(A) = \mathbb{R}^n$
$\Leftrightarrow$ the number of pivot columns in the row echelon form of A is n.

Solutions to Exercises

1. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

2. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

3. Since $T(\mathbf{v}) = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$, $\mathbf{v}$ is not in $N(T)$.

4. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

5. Since $p'(x) = 2x - 3$ and $p''(x) = 2$, then $T(p(x)) = 2x$, so $p(x)$ is not in $N(T)$.

6. Since $p'(x) = 5$ and $p''(x) = 0$, then $T(p(x)) = 0$, so $p(x)$ is in $N(T)$.

7. Since $T(p(x)) = -2x$, then $p(x)$ is not in $N(T)$.

8. Since $T(p(x)) = 0$, then $p(x)$ is in $N(T)$.

9. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ there are infinitely many vectors that are mapped to $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$. For example, $T\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and hence, $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is in $R(T)$.

10. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 4 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & \\ 0 & 0 & 0 & 1 \end{array}\right]$ the linear system is inconsistent, so the vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is not in $R(T)$.

11. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & -1 & 3 & -2 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, the linear system is inconsistent, so the vector $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ is not in $R(T)$.

12. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 2 & 1 & 3 & -5 \\ 1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ there are infinitely many vectors that are mapped to $\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$ and hence, the vector $\begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$ is in $R(T)$.

13. The matrix A is in $R(T)$.

14. The matrix A is not in $R(T)$.

15. The matrix A is not in $R(T)$.

16. The matrix A is in $R(T)$.

17. A vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in the null space, if and only if $3x + y = 0$ and $y = 0$. That is, $N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. Hence, the null space has dimension 0, so does not have a basis.

18. A vector is in the null space if and only if $\begin{cases} -x + y & = 0 \\ x - y & = 0 \end{cases}$, that is $x = y$. Therefore, $N(T) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$ and hence, a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

19. Since $\begin{bmatrix} x + 2z \\ 2x + y + 3z \\ x - y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if and only if $x = -2z$ and $y = z$ every vector in the null space has the form $\begin{bmatrix} -2z \\ z \\ z \end{bmatrix}$. Hence, a basis for the null space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

20. Since $\begin{bmatrix} -2 & 2 & 2 \\ 3 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$, then $N(T) = \left\{ t \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$ and a basis for the null space is $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$.

21. Since $N(T) = \left\{ \begin{bmatrix} 2s + t \\ s \\ t \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$, a basis for the null space is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

22. A basis for the null space is $\left\{ \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Exercise Set 4.3

An isomorphism between vector spaces establishes a one-to-one correspondence between the vector spaces. If $T : V \longrightarrow W$ is a one-to-one and onto linear transformation, then T is called an isomorphism. A mapping is one-to-one if and only if $N(T) = \{\mathbf{0}\}$ and is onto if and only if $R(T) = W$. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis for V and $T : V \longrightarrow W$ is a linear transformation, then $R(T) = \textbf{span}\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$. If in addition, T is one-to-one, then $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ is a basis for $R(T)$. The main results of Section 4.3 are:

- If V is a vector space with $\dim(V) = n$, then V is isomorphic to $\mathbb{R}^n$.
- If V and W are vector spaces of dimension n, then V and W are isomorphic.

For example, there is a correspondence between the very different vector spaces $\mathcal{P}_3$ and $M_{2\times 2}$. To define the isomorphism, start with the standard basis $S = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$. Since every polynomial $a+bx+cx^2+dx^3 = a(1) + b(x) + c(x^2) + d(x^3)$ use the coordinate map

$$a + bx + cx^2 + dx^3 \xrightarrow{L_1} [a + bx + cx^2 + dx^3]_S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ followed by } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

so that the composition $L_2(L_1(a + bx + cx^2 + dx^3)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines an isomorphism between $\mathcal{P}_3$ and $M_{2\times 2}$.

Solutions to Exercises

1. Since $N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, then T is one-to-one.

2. Since $N(T) = \left\{ \begin{bmatrix} -a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$, then T is not one-to-one.

3. Since $N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, then T is one-to-one.

4. Since $\begin{bmatrix} 2 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -4 & -1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & -2 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 3 \end{bmatrix}$, then $N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, so T is one-to-one.

5. Let $p(x) = ax^2 + bx + c$, so that $p'(x) = 2ax + b$. Then

$$T(p(x)) = 2ax + b - ax^2 - bx - c = -ax^2 + (2a - b)x + (b - c) = 0$$

if and only if $a = 0, 2a - b = 0, b - c = 0$. That is, $p(x)$ is in $N(T)$ if and only if $p(x) = 0$. Hence, T is one-to-one.

6. Let $p(x) = ax^2 + bx + c$, so $T(p(x)) = ax^3 + bx^2 + cx = 0$ if and only if $a = b = c = 0$. Therefore, $N(T)$ consists of only the zero polynomial and hence, T is one-to-one.

7. A vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the range of T if the linear system $\begin{cases} 3x - y &= a \\ x + y &= b \end{cases}$ has a solution. Since the linear system is consistent for every vector $\begin{bmatrix} a \\ b \end{bmatrix}$, T is onto $\mathbb{R}^2$. Notice the result also follows from $\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\right) = 4$, so the inverse exists.

8. Since $\left[\begin{array}{cc|c} -2 & 1 & a \\ 1 & -1/2 & b \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} -2 & 1 & a \\ 0 & 0 & \frac{1}{2}a + b \end{array}\right]$, then a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the range of T if and only if $a = -2b$ and hence, T is not onto.

9. Since $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ is row equivalent to the identity matrix, then the linear operator T is onto $\mathbb{R}^3$.

10. Since

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & a \\ -1 & 1 & 3 & b \\ 1 & 4 & 2 & c \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 2 & 3 & -1 & a \\ 0 & 5 & 5 & a+2b \\ 0 & 0 & 0 & -a-b+c \end{array}\right],$$

then a vector is in the range of T if and only if $-a-b+c=0$ and hence, T is not onto.

11. Since $T(\mathbf{e_1}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $T(\mathbf{e_2}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ are two linear independent vectors in $\mathbb{R}^2$, they form a basis.

12. Since $T(\mathbf{e_2}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the set is not a basis.

13. Since $T(\mathbf{e_1}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and $T(\mathbf{e_2}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ are two linear independent vectors in $\mathbb{R}^2$, they form a basis.

14. Since $T(\mathbf{e_2}) = 2T(\mathbf{e_1})$, the set is not a basis.

15. Since $T(\mathbf{e_1}) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $T(\mathbf{e_2}) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $T(\mathbf{e_3}) = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ are three linear independent vectors in $\mathbb{R}^3$, they form a basis.

16. Since $\begin{vmatrix} 2 & 3 & -1 \\ 2 & 6 & 3 \\ 4 & 9 & 2 \end{vmatrix} = 0$, the set is linearly dependent and hence, is not a basis.

17. Since $\begin{vmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -1 & 3/2 \end{vmatrix} = 4$, the set is linearly independent and hence, is a basis.Is a basis.

18. Since $\begin{vmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 0 & -1 & 5 \end{vmatrix} = 6$, the set is linearly independent and hence, is a basis.

19. Since $T(1) = x^2, T(x) = x^2 - 3x$, and $T(x^2) = x^2 + x + 1$, are three linearly independent polynomials the set is a basis.

20. Since $T(1) = 0$, the set is not a basis.

21. **a.** Since $\det(A) = \det\left(\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}\right) = -3 \neq 0$, then the matrix A is invertible and hence, T is an isomorphism. **b.** $A^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$ **c.** Let $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. To show that $T^{-1}(\mathbf{w}) = A^{-1}\mathbf{w}$, we will show that $A^{-1}(T(\mathbf{w})) = \mathbf{w}$. That is,

$$A^{-1}T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -2/3 & -1/3 \end{bmatrix}\begin{bmatrix} x \\ -2x-3y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

22. **a.** Since $\det(A) = \det\left(\begin{bmatrix} -2 & 3 \\ -1 & -1 \end{bmatrix}\right) = 5 \neq 0$, then the matrix A is invertible and hence, T is an isomorphism. **b.** $A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}$ **c.** Let $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. To show that $T^{-1}(\mathbf{w}) = A^{-1}\mathbf{w}$, we will show that $A^{-1}(T(\mathbf{w})) = \mathbf{w}$. That is,

$$A^{-1}T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2x+3y \\ -x-y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5x \\ 5y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

- The coordinates of any vector $T(\mathbf{v})$ can be found using the matrix product

$$[T(\mathbf{v}))]_{B'} = [T]_B^{B'}[\mathbf{v}]_B.$$

- As an example, let $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$, then after applying the operator T the coordinates relative to B' is given by

$$\left[T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 1 & 1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}\left[\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right]_B.$$

Since B is the standard basis the coordinates of a vector are just the components, so

$$\left[T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 1 & 1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -9/2 \\ 3/2 \end{bmatrix}.$$

This vector is not $T(\mathbf{v})$, but the coordinates relative to the basis B'. Then

$$T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -7/2 \end{bmatrix}.$$

Other useful formulas that involve combinations of linear transformations and the matrix representation are:

• $[S+T]_B^{B'} = [S]_B^{B'}+[T]_B^{B'}$ • $[kT]_B^{B'} = k[T]_B^{B'}$ • $[S \circ T]_B^{B'} = [S]_B^{B'}[T]_B^{B'}$ • $[T^n]_B = ([T]_B)^n$ • $[T^{-1}]_B = ([T]_B)^{-1}$

Solutions to Exercises

1. a. Let $B = \{\mathbf{e_1}, \mathbf{e_2}\}$ be the standard basis. To find the matrix representation for A relative to B, the column vectors are the coordinates of $T(\mathbf{e_1})$ and $T(\mathbf{e_2})$ relative to B. Recall the coordinates of a vector relative to the standard basis are just the components of the vector. Hence, $[T]_B = [\ [T(\mathbf{e_1})]_B\ [T(\mathbf{e_2})]_B\ ] = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$.
b. The direct computation is $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ and using part (a), the result is $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$.

2. a. $[T]_B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b.** The direct computation is $T\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and using part (a), the result is $T\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

3. a. Let $B = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis. Then $[T]_B = [\ [T(\mathbf{e_1})]_B\ [T(\mathbf{e_2})]_B\ [T(\mathbf{e_2})]_B\ ] = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$. **b.** The direct computation is $T\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$, and using part (a) the result is $T\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$.

31. Since $[T]_B = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $[T(p(x)]_B = \begin{bmatrix} -12 \\ -48 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $T(p(x)) = p'''(x) = -12 - 48x$.

32. Since B is the standard basis, then $[T(1)]_B = [1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(x)]_B = [2x]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $[T(x^2)]_B = [3x^2]_B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, so $[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

33. $[S]_B^{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[D]_B^{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $[D]_B^{B'}[S]_B^{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = [T]_B$

34. The linear operator that reflects a vector through the line perpendicular to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is reflects across the line $y = -x$, is given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$, so

$$[T]_B = \left[\begin{bmatrix} -1 \\ -1 \end{bmatrix}_B \begin{bmatrix} -1 \\ 0 \end{bmatrix}_B \right] = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

35. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the matrix representation for T is $[T]_S = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ -c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{bmatrix}$.

36. Since $T(\mathbf{v}) = \mathbf{v}$ is the identity map, then

$$[T]_B^{B'} = [\ [T(\mathbf{v_1})]_{B'}\ [T(\mathbf{v_2})]_{B'}\ [T(\mathbf{v_3})]_{B'}] = [\ [\mathbf{v_1}]_{B'}\ [\mathbf{v_2}]_{B'}\ [\mathbf{v_3}]_{B'}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $[\mathbf{v}]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then $[\mathbf{v}]_{B'} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$. The matrix $[T]_B^{B'}$ can be obtained from the identity matrix by interchanging the first and second columns.

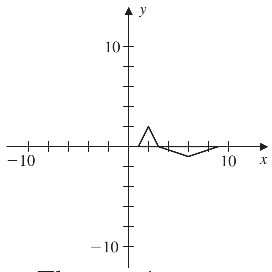
37.

$$[T]_B = [\ [T(\mathbf{v_1})]_B\ [T(\mathbf{v_2})]_B \ldots [T(\mathbf{v_n})\ ]_B = [\ [\mathbf{v_1}]_B\ [\mathbf{v_1}+\mathbf{v_2}]_B \ldots [\mathbf{v_{n-1}}+\mathbf{v_n}\ ]_B\ ] = \begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

Exercise Set 4.5

If $T : V \longrightarrow V$ is a linear operator the matrix representation of T relative to a basis B, denoted $[T]_B$

b.

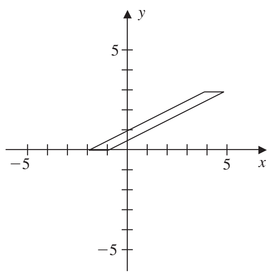


c. The matrix that will reverse the action of the operator T is the inverse of $[T]_S$. That is, $[T]_S^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & -2 \end{bmatrix}$.

4. a. The matrix representation relative to the standard basis S is the product of the matrix representations for the three separate operators. That is,

$$[T]_S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

b.

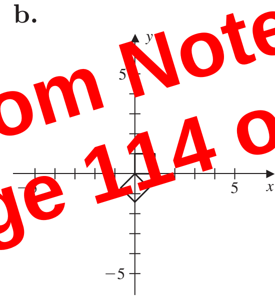


c. The matrix that will reverse the action of the operator T is the inverse of $[T]_S$. That is, $[T]_S^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$.

5. a.

$$[T]_S = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

b.



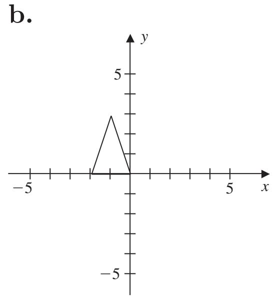
c.

$$[T]_S^{-1} = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

6. a.

$$[T]_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

b.



c.

$$[T]_S^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

d. The transformation is a reflection through the y-axis.

7. a. $[T]_S = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & \sqrt{3}/2 - 1/2 \\ 1/2 & \sqrt{3}/2 & \sqrt{3}/2 + 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

9. Since $T^2 - T + I = \mathbf{0}, T - T^2 = I$. Then

$$(T \circ (I - T))(\mathbf{v}) = T((I-T)(\mathbf{v})) = T(\mathbf{v} - T(\mathbf{v})) = T(\mathbf{v}) - T^2(\mathbf{v}) = I(\mathbf{v}) = \mathbf{v}.$$

10. a. The point-slope equation of the line that passes through the points given by $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is $y = \frac{v_2 - u_2}{v_1 - u_1}(x - u_1) + u_2$. Now consider $t\mathbf{u} + (1-t)\mathbf{v} = \begin{bmatrix} tu_1 + (1-t)v_1 \\ tu_2 + (1-t)v_2 \end{bmatrix}$ and show the components satisfy the point-slope equation. That is,

$$\begin{aligned} \frac{v_2 - u_2}{v_1 - u_1}(tu_1 + (1-t)v_1 - u_1) + u_2 &= \frac{v_2 - u_2}{v_1 - u_1}(t(u_1 - v_1) + ((v_1 - u_1)) + u_2 \\ &= (v_2 - u_2)(1-t) + u_2 = (1-t)v_2 - u_2(1-t) + u_2 \\ &= tu_2 + (1-t)v_2. \end{aligned}$$

b. Since $T(t\mathbf{u} + (1-t)\mathbf{v}) = tT(\mathbf{u}) + (1-t)T(\mathbf{v})$, then the image of a line segment is another line segment. **c.** Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be two vectors in $T(S)$. Since T is one-to-one and onto there are unique vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ in S such that $T(\mathbf{v_1}) = \mathbf{w_1}$ and $T(\mathbf{v_2}) = \mathbf{w_2}$. Since S is a convex set for $0 \le t \le 1$, we have $t\mathbf{v_1} + (1-t)\mathbf{v_2}$ is in S and hence, $T(t\mathbf{v_1} + (1-t)\mathbf{v_2})$ is in S. But $T(t\mathbf{v_1} + (1-t)\mathbf{v_2}) = tT(\mathbf{v_1}) + (1-t)T(\mathbf{v_2}) = t\mathbf{w_1} + (1-t)\mathbf{w_2}$, which is in $T(S)$ and hence, $T(S)$ is a convex set. **d.** Since T is one-to-one and onto the linear transformation is an isomorphism. To find the image of S, let $\begin{bmatrix} x \\ y \end{bmatrix}$ be a vector in S and let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ so that $u = 2x$, and $v = y$. Then $\left(\frac{u}{2}\right)^2 + v^2 = x^2 + y^2 = 1$. Therefore, $T(S) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \middle| \frac{u^2}{4} + v^2 = 1 \right\}$, which is an ellipse in $\mathbb{R}^2$.

Chapter Test Chapter 4

1. F.

$$T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$$

since the second component of the sum will contain a plus 4.

2. F. Since

$$T(x + y) = 2x + 2y - 1$$

but

$$T(x) + T(y) = 2x + 2y - 2.$$

3. T

4. T

5. T

6. F. Since

$$\begin{aligned} T(\mathbf{u}) &= \frac{1}{3}T(2\mathbf{u} - \mathbf{v}) + \frac{1}{3}T(\mathbf{u} + \mathbf{v}) \\ &= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}. \end{aligned}$$

7. F. Since

$$N(T) = \left\{ \begin{bmatrix} 2t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

8. F. If T is one-to-one, then the set is linearly independent.

9. T

10. F. For example, $T(1) = 0 = T(2)$.

11. T

12. F. Since, for every k, $T\left(\begin{bmatrix} k \\ k \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

13. T

14. T

15. T

16. T

17. T

18. T

5 Eigenvalues and Eigenvectors

Exercise Set 5.1

An eigenvalue of the $n \times n$ matrix A is a number λ such that there is a nonzero vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$. So if λ and $\mathbf{v}$ are an eigenvalue–eigenvector pair, then the action on $\mathbf{v}$ is a scaling of the vector. Notice that if $\mathbf{v}$ is an eigenvalue corresponding to the eigenvalue λ, then

$$A(c\mathbf{v}) = cA\mathbf{v} = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v}),$$

so A will have infinitely many eigenvectors corresponding to the eigenvalue λ. Also recall that an eigenvalue can be 0 (or a complex number), but eigenvectors are only nonzero vectors. An eigenspace is the set of all eigenvectors corresponding to an eigenvalue λ along with the zero vector, and is denoted by $V_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$. Adding the zero vector makes V_λ a subspace. The eigenspace can also be viewed as the null space of $A - \lambda I$. To determine the eigenvalues of a matrix A we have:

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow \det(A - \lambda I) = 0.$$

The last equation is the characteristic equation for A. As an immediate consequence, if A is a triangular matrix, then the eigenvalues are the entries on the diagonal. To then find the corresponding eigenvectors, for each eigenvalue λ, the equation $A\mathbf{v} = \lambda\mathbf{v}$ is solved for $\mathbf{v}$. An outline of the typical computations for the matrix $A = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ are:

- To find the eigenvalues solve the equation $\det(A - \lambda I) = 0$. Expanding across row three, we have that

$$\begin{vmatrix} -1-\lambda & 1 & -2 \\ 1 & -1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1-\lambda & 2 \end{vmatrix} + (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = -\lambda^2 - \lambda^3.$$

  Then $\det(A - \lambda I) = 0$ if and only if

$$-\lambda^2 - \lambda^3 = -\lambda^2(1+\lambda) = 0, \text{ so the eigenvalues are } \lambda_1 = 0, \lambda_2 = 0.$$

- To find the eigenvectors corresponding to $\lambda_1 = 0$ solve $A\mathbf{v} = \mathbf{0}$. Since

$$\begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

  the eigenvectors are of the form $\begin{bmatrix} -t \\ t \\ t \end{bmatrix}$, for any $t \neq 0$. Similarly, the eigenvectors of $\lambda_2 = -1$ have the from $\begin{bmatrix} -2t \\ 2t \\ t \end{bmatrix}$, $t \neq 0$.

- The eigenspaces are $V_0 = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R}^3 \right\}$ and $V_{-1} = \left\{ t \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R}^3 \right\}$.

- Notice that there are only two linearly independent eigenvectors of A, the algebraic multiplicity of $\lambda_1 = 0$ is 2, the algebraic multiplicity of $\lambda_2 = -1$ is 1, and the geometric multiplicities are both 1. For a 3×3 matrix other possibilities are:

Exercise Set 5.3

1. The strategy is to uncouple the system of differential equations. Writing the system in matrix form, we have that

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{y}.$$

The next step is to diagonalize the matrix A. Since A is triangular the eigenvalues of A are the diagonal entries -1 and -2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. So $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$. The related uncoupled system is $\mathbf{w}' = P^{-1}AP\mathbf{w} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{w}$. The general solution to the uncoupled system is $\mathbf{w}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \mathbf{w}(0)$. Finally, the general solution to the original system is given by $\mathbf{y}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1}\mathbf{y}(0)$. That is,

$$y_1(t) = (y_1(0) + y_2(0))e^{-t} - y_2(0)e^{-2t}, y_2(t) = y_2(0)e^{-2t}.$$

2. Let $A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$. Then the eigenvalues of A are 1 and -2 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively. So $A = PDP^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}$ and hence, $\mathbf{w}'(t) = P^{-1}AP\mathbf{w} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$. Then the general solution to the uncoupled system is $\mathbf{w}(t) = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \mathbf{w}(0)$ and hence $\mathbf{y}(t) = P \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1}\mathbf{y}(0)$, that is,

$$y_1(t) = \frac{1}{3}(y_1(0) + 2y_2(0))e^{t} + \frac{2}{3}(y_1(0) - y_2(0))e^{-2t}, y_2(t) = \frac{1}{3}(y_1(0) + 2y_2(0))e^{t} + \frac{1}{3}(-y_1(0) + y_2(0))e^{-2t}.$$

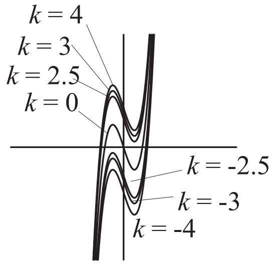
3. Using the same approach as in Exercise (1), we let $A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$. The eigenvalues of A are 4 and -2 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively, so that
$A = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. So the general solution is given by $\mathbf{y}(t) = P \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1}\mathbf{y}(0)$, that is

$$y_1(t) = \frac{1}{2}(y_1(0) - y_2(0))e^{4t} + \frac{1}{2}(y_1(0) + y_2(0))e^{-2t}, \quad y_2(t) = \frac{1}{2}(-y_1(0) + y_2(0))e^{4t} + \frac{1}{2}(y_1(0) + y_2(0))e^{-2t}.$$

4. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then the eigenvalues of A are 0 and 2 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. So $A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$ and hence, $\mathbf{w}'(t) = P^{-1}AP\mathbf{w} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}$. Then the general solution to the uncoupled system is $\mathbf{w}(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{w}(0)$ and hence $\mathbf{y}(t) = P \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{y}(0)$, that is,

$$y_1(t) = \frac{1}{2}(y_1(0) + y_2(0)) + \frac{1}{2}(y_1(0) - y_2(0))e^{2t}, y_2(t) = \frac{1}{2}(y_1(0) + y_2(0)) + \frac{1}{2}(-y_1(0) + y_2(0))e^{2t}.$$

b. The graphs of $y(\lambda) = \lambda^3 - 3\lambda + k$ for different values of k are shown in the figure.



c. The matrix will have three distinct real eigenvalues when the graph of $y(\lambda) = \lambda^3 - 3\lambda + k$ crosses the x-axis three time. That is, for $-2 < k < 2$.

6. Since $B = P^{-1}AP$, we have that $A = PBP^{-1}$. Suppose $B\mathbf{v} = \lambda\mathbf{v}$, so $\mathbf{v}$ is an eigenvector of B corresponding to the eigenvalue λ. Then

$$A(P\mathbf{v}) = PBP^{-1}P\mathbf{v} = PB\mathbf{v} = P(\lambda\mathbf{v}) = \lambda P\mathbf{v}$$

and hence, $P\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

7. a. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Then each component of the vector $A\mathbf{v}$ has the same value equal to the common row sum λ. That is, $A\mathbf{v} = \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, so λ is an eigenvalue of A corresponding to the eigenvector $\mathbf{v}$. **b.** Since A and A^t have the same eigenvalues, then the same result holds if the sum of each column of A is equal to λ.

8. a. Since T is a linear operator $T(\mathbf{0}) = \mathbf{0}$, so $\{\mathbf{0}\}$ is invariant. And for every $\mathbf{v}$ in V, then $T(\mathbf{v})$ is in V so V is invariant. **b.** Since $\dim(W) = 1$, there is a nonzero vector $\mathbf{w_0}$ such that $W = \{a\mathbf{w_0} \mid a \in \mathbb{R}\}$. Then $T(\mathbf{w_0}) = \mathbf{w_1}$ and since W is invariant, $\mathbf{w_0}$ is in W. So there is some λ such that $\mathbf{w_1} = \lambda\mathbf{w_0}$. Hence, $\mathbf{w_0}$ is an eigenvector of T. **c.** By part (a), $\mathbb{R}^2$ and $\{\mathbf{0}\}$ are invariant subspaces of the linear operator T. Since the matrix representation for T is given relative to the standard basis,

$$T(\mathbf{v}) = \lambda\mathbf{v} \Leftrightarrow T(\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow v_1 = v_2 = 0.$$

By part (b), the only invariant subspaces of T are $\mathbb{R}^2$ and $\{\mathbf{0}\}$.

9. a. Suppose $\mathbf{w}$ is in $S(V_{\lambda_0})$, so that $\mathbf{w} = S(\mathbf{v})$ for some eigenvector $\mathbf{v}$ of T corresponding to λ_0. Then

$$T(\mathbf{w}) = T(S(\mathbf{v})) = S(T(\mathbf{v})) = S(\lambda_0\mathbf{v}) = \lambda_0 S(\mathbf{v}) = \lambda_0\mathbf{w}.$$

Hence, $S(V_{\lambda_0}) \subseteq V_{\lambda_0}$.

b. Let $\mathbf{v}$ be an eigenvector of T corresponding to the eigenvalue λ_0. Since T has n distinct eigenvalues then $\dim(V_{\lambda_0}) = 1$ with $V_{\lambda_0} = \mathbf{span}\{\mathbf{v}\}$. Now by part (a), $T(S(\mathbf{v})) = \lambda_0(S(\mathbf{v}))$, so that $S(\mathbf{v})$ is also an eigenvector of T and in $\mathbf{span}\{\mathbf{v}\}$. Consequently, there exists a scalar μ_0 such that $S(\mathbf{v}) = \mu_0\mathbf{v}$, so that $\mathbf{v}$ is also an eigenvector of S.
c. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis for V consisting of eigenvectors of T and S. Thus there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ such that $T(\mathbf{v_i}) = \lambda_i\mathbf{v_i}$ and $S(\mathbf{v_i}) = \mu_i\mathbf{v_i}$, for $1 \leq i \leq n$. Now let $\mathbf{v}$ be a vector in V. Since B is a basis for V then there are scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}$. Applying the operator ST to both sides of this equation we obtain

$$\begin{aligned} ST(\mathbf{v}) &= ST(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}) = S(c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} + \ldots + c_n\lambda_n\mathbf{v_n}) \\ &= c_1\lambda_1\mu_1\mathbf{v_1} + c_2\lambda_2\mu_2\mathbf{v_2} + \ldots + c_n\lambda_n\mu_n\mathbf{v_n} = c_1\mu_1\lambda_1\mathbf{v_1} + c_2\mu_2\lambda_2\mathbf{v_2} + \ldots + c_n\mu_n\lambda_n\mathbf{v_n} \\ &= T(c_1\mu_1\mathbf{v_1} + c_2\mu_2\mathbf{v_2} + \ldots + c_n\mu_n\mathbf{v_n}) = TS(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}) = TS(\mathbf{v}). \end{aligned}$$

Solutions to Exercises

1. $\mathbf{u}\cdot\mathbf{v} = (0)(1) + (1)(-1) + (3)(2) = 5$

2. $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} = \frac{0-1+6}{1+1+4} = \frac{5}{6}$

3.

$$\mathbf{u}\cdot(\mathbf{v}+2\mathbf{w}) = \mathbf{u}\cdot\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 0 + 1 - 12 = -11$$

4. $\frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{0+1-9}{1+1+9}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = -\frac{8}{11}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

5. $\| \mathbf{u} \| = \sqrt{1^2 + 5^2} = \sqrt{26}$

6. $||\mathbf{u}-\mathbf{v}|| = \sqrt{\begin{bmatrix} -1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 4 \end{bmatrix}} = \sqrt{17}$

7. Divide each component of the vector by the norm of the vector, so that $\frac{1}{\sqrt{26}}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is a unit vector in the direction of $\mathbf{u}$.

8. Since $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{7}{\sqrt{26}\sqrt{5}}$, then the vectors are not orthogonal.

9. $\frac{10}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

10. The vector $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ if and only if $w_1 + 5w_2 = 0$ and $2w_1 + w_2 = 0$, that is, $w_1 = 0 = w_2$.

11. $\| \mathbf{u} \| = \sqrt{(-3)^2 + (-2)^2 + 3^2} = \sqrt{22}$

12. $||\mathbf{u}-\mathbf{v}|| = \sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}$

$$= \sqrt{\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}} = \sqrt{41}$$

13. $\frac{1}{\sqrt{22}}\begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$

14. Since $\cos\theta = -\frac{4}{11\sqrt{2}} \neq 0$, then the vectors are not orthogonal.

15. $-\frac{3}{\sqrt{11}}\begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} = \frac{3}{\sqrt{11}}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

16. A vector $\mathbf{w}$ is orthogonal to both vectors if and only if $\begin{cases} -3w_1 - 2w_2 + 3w_3 &= 0 \\ -w_1 - w_2 - 3w_3 &= 0 \end{cases} \Leftrightarrow$ $w_1 = 9w_3, w_2 = -12w_3$. So all vectors in $\textbf{span}\left\{\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}\right\}$ are orthogonal to the two vectors.

17. Since two vectors in $\mathbb{R}^2$ are orthogonal if and only if their dot product is zero, solving $\begin{bmatrix} c \\ 3 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$, gives $-c + 6 = 0$, that is, $c = 6$.

18. $\begin{bmatrix} -1 \\ c \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = 0 + 2c - 2 = 0 \Leftrightarrow c = 1$

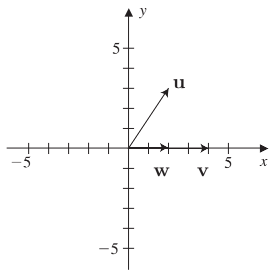
19. The pairs of vectors with dot product equaling 0 are $\mathbf{v_1}\perp\mathbf{v_2}$, $\mathbf{v_1}\perp\mathbf{v_4}$, $\mathbf{v_1}\perp\mathbf{v_5}$, $\mathbf{v_2}\perp\mathbf{v_3}$, $\mathbf{v_3}\perp\mathbf{v_4}$, and $\mathbf{v_3}\perp\mathbf{v_5}$.

20. The vectors $\mathbf{v_2}$ and $\mathbf{v_5}$ are in the same direction.

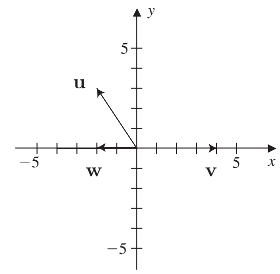
21. Since $\mathbf{v_3} = -\mathbf{v_1}$, the vectors $\mathbf{v_1}$ and $\mathbf{v_3}$ are in opposite directions.

22. $||\mathbf{v_4}|| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$

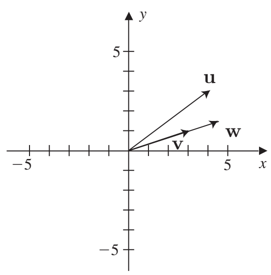
23. $\mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$



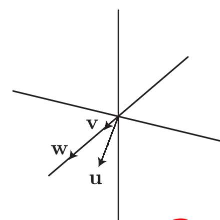
24. $\mathbf{w} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



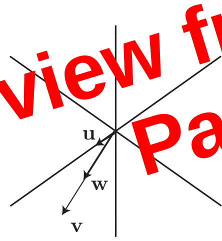
25. $\mathbf{w} = \frac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$



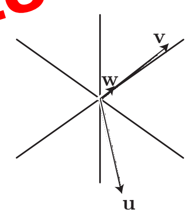
26. $\mathbf{w} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$



27. $\mathbf{w} = \frac{1}{6}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$



28. $\mathbf{w} = \frac{3}{13}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$



29. Let $\mathbf{u}$ be a vector in $\mathbf{span}\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$. Then there exist scalars $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{u} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}.$$

Using the distributive property of the dot product gives

$$\begin{aligned}\mathbf{v}\cdot\mathbf{u} &= \mathbf{v}\cdot(c_1\mathbf{u_1} + c_2\mathbf{u_2} + \ldots + c_n\mathbf{u_n}) \\ &= c_1(\mathbf{v}\cdot\mathbf{u_1}) + c_2(\mathbf{v}\cdot\mathbf{u_2}) + \cdots + c_n(\mathbf{v}\cdot\mathbf{u_n}) \\ &= c_1(0) + c_2(0) + \cdots + c_n(0) = 0.\end{aligned}$$

30. If $\mathbf{u}$ and $\mathbf{w}$ are in S and c is a scalar, then

$$(\mathbf{u} + c\mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + c(\mathbf{w}\cdot\mathbf{v}) = 0 + 0 = 0$$

and hence, S is a subspace.

19. An orthonormal basis for $\mathbf{span}(W)$ is

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

20. An orthonormal basis for $\mathbf{span}(W)$ is

$$\left\{ \frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{\sqrt{3}}{3} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

21. An orthonormal basis for $\mathbf{span}(W)$ is

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

22. An orthonormal basis for $\mathbf{span}(W)$ is

$$\left\{ \frac{\sqrt{5}}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \frac{\sqrt{15}}{15} \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, -\frac{\sqrt{30}}{30} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 5 \end{bmatrix} \right\}.$$

23. An orthonormal basis for $\mathbf{span}(W)$ is

$$\left\{ \sqrt{3}x, \ -3x+2 \right\}.$$

24. An orthogonal basis for $\mathbf{span}(W)$ is

$$\left\{ 1, 12x-6, -\frac{5}{2}x^3 + \frac{9}{4}x - \frac{1}{2} \right\}$$

and an orthonormal basis is

$$\left\{ 1, \frac{1}{\sqrt{12}}(12x-6), \frac{4\sqrt{7}}{3}\left(-\frac{5}{2}x^3 + \frac{9}{4}x - \frac{1}{2}\right) \right\}.$$

25. An orthonormal basis for $\mathbf{span}(W)$ is $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$

26. An orthonormal basis for $\mathbf{span}(W)$ is $\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \right\}.$

27. Let $\mathbf{v}$ be a vector in V and $B = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ an orthonormal basis for V. Then there exist scalars $c_1, \ c_2, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}$. Then

$$||\mathbf{v}||^2 = \mathbf{v}\cdot\mathbf{v} = c_1^2(\mathbf{u_1}\cdot\mathbf{u_1}) + c_2^2(\mathbf{u_2}\cdot\mathbf{u_2}) + \cdots + c_n^2(\mathbf{u_n}\cdot\mathbf{u_n}).$$

Since B is orthonormal each vector in B has norm one, $1 = ||\mathbf{u_i}||^2 = \mathbf{u_i}\cdot\mathbf{u_i}$ and they are pairwise orthogonal, so $\mathbf{u_1}\cdot\mathbf{u_j} = 0$, for $i \neq j$. Hence,

$$||\mathbf{v}||^2 = c_1^2 + c_2^2 + \cdots + c_n^2 = |\mathbf{v}\cdot\mathbf{u_1}|^2 + \cdots + |\mathbf{v}\cdot\mathbf{u_n}|^2.$$

28. To show the three statements are equivalent we will show that (a)⇒(b), (b)⇒(c) and (c)⇒(a).

- (a)⇒(b): Suppose that $A^{-1} = A^t$. Since $AA^t = I$, then the row vectors are orthonormal. Since they are orthogonal they are linearly independent and hence a basis for $\mathbb{R}^n$.
- (b)⇒(c): Suppose the row vectors of A are orthonormal. Then $A^tA = I$ and hence the column vectors of A are orthonormal.
- (c)⇒(a): Suppose the column vectors are orthonormal. Then $AA^t = I$ and hence, A is invertible with $A^{-1} = A^t$.

The vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is in $W^{\perp}$ if and only if it is orthogonal to each of the three vectors that generate W, so that $x - w = 0, y - w = 0, z - w = 0, z \in \mathbb{R}$. Hence, a basis for the orthogonal complement of W is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

16. The two vectors that span W are linearly independent but are not orthogonal. Using the Gram-Schmidt process an orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix} \right\}$. Then

$$\mathbf{proj}_W \mathbf{v} = \frac{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix}}{\begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix}} \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 2 \\ 5 \\ 13 \end{bmatrix}.$$

17. The two vectors that span W are linearly independent and orthogonal, so that an orthogonal basis for W is $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$. Then

$$\mathbf{proj}_W \mathbf{v} = \frac{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{1} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

18. The two vectors that span W are linearly independent but not orthogonal. Using the Gram-Schmidt process an orthogonal basis for W is $B = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 14 \\ 8 \end{bmatrix} \right\}$. Then

$$\mathbf{proj}_W \mathbf{v} = \frac{\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 14 \\ 8 \end{bmatrix}}{\begin{bmatrix} 2 \\ 14 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 14 \\ 8 \end{bmatrix}} \begin{bmatrix} 2 \\ 14 \\ 8 \end{bmatrix} = \frac{19}{11} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 1 \\ 14 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

Observe that $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ is contained in the subspace W.

Let $\mathbf{v_1} = \begin{bmatrix} a \\ b \end{bmatrix}$ and let θ be the angle that $\mathbf{v_1}$ makes with the horizontal axis. Since $a^2 + b^2 = 1$, then $\mathbf{v_1}$ is a unit vector. Therefore $a = \cos\theta$ and $b = \sin\theta$. Now let $\mathbf{v_2} = \begin{bmatrix} c \\ d \end{bmatrix}$. Since $ac + bd = 0$ then $\mathbf{v_1}$ and $\mathbf{v_2}$ are orthogonal. There are two cases.

Case 1. $c = \cos\theta + \pi/2 = -\sin\theta$ and $d = \sin\theta + \pi/2 = \cos\theta$, so that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Case 2. $c = \cos\theta - \pi/2 = \sin\theta$ and $d = \sin\theta - \pi/2 = -\cos\theta$, so that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

c. If $\det(A) = 1$, then by part (b), $T(\mathbf{v}) = A\mathbf{v}$ with $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Therefore

$$T(\mathbf{v}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which is a rotation of a vector by $-\theta$ radians. If $\det(A) = -1$, then by part (b), $T(\mathbf{v}) = A'\mathbf{v}$ with $A' = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Observe that

$$A' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence, in this case, T is a reflection through the x-axis followed by a rotation through the angle θ.

28. Suppose A and B are orthogonally similar, so $B = P^tAP$, where P is an orthogonal matrix. Since P is orthogonal $P^{-1} = P^t$.

a. First suppose A is symmetric, so $A^t = A$. Then $B^t = (P^tAP)^t = P^tA^tP = P^tAP = B$ and hence, B is symmetric. Conversely, suppose B is symmetric, so $B = B^t$. Since $B = P^tAP = P^{-1}AP$, then $A = PBP^{-1}$. Then $A^t = (PBP^{-1})^t = (PBP^t)^t = PB^tP^t = PBP^t = A$ and hence A is symmetric.

b. First suppose A is orthogonal, so $A^{-1} = A^t$. Then $B^{-1} = (P^tAP)^{-1} = P^{-1}A^{-1}P = P^tA^tP = (P^tAP)^t = B^t$ and hence, B is orthogonal. Conversely, suppose B is orthogonal, so $B^{-1} = B^t$. Since $B = P^tAP = P^{-1}AP$, then $A = PBP^{-1}$. Then $A^{-1} = (PBP^{-1})^{-1} = PB^{-1}P^{-1} = PB^tP^t = (P^tBP)^t = A^t$ and hence, A is orthogonal.

29. Suppose $D = P^tAP$, where P is an orthogonal matrix, that is $P^{-1} = P^t$. Then

$$D^t = (P^tAP)^t = P^tA^tP.$$

Since D is a diagonal matrix then $D^t = D$, so we also have $D = P^tA^tP$ and hence, $P^tAP = P^tA^tP$. Then $P(P^tAP)P^t = P(P^tA^tP)P^t$. Since $PP^t = I$, we have that $A = A^t$, and hence, the matrix A is symmetric.

30. Suppose A^{-1} exists and $D = P^tAP$, where D is a diagonal matrix and P is orthogonal. Since $D = P^tAP = P^{-1}AP$, then $A = PDP^{-1}$. Then $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$, so $D^{-1} = P^{-1}A^{-1}P = P^tA^{-1}P$ and hence A^{-1} is orthogonally diagonalizable.

31. a. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then $\mathbf{v}^t\mathbf{v} = v_1^2 + \ldots + v_n^2$.

b. Consider the equation $A\mathbf{v} = \lambda\mathbf{v}$. Now take the transpose of both sides to obtain $\mathbf{v}^tA^t = \lambda\mathbf{v}^t$. Since A is skew symmetric this is equivalent to

$$\mathbf{v}^t(-A) = \lambda\mathbf{v}^t.$$

Now, right multiplication of both sides by $\mathbf{v}$ gives $\mathbf{v}^t(-A\mathbf{v}) = \lambda\mathbf{v}^t\mathbf{v}$ or equivalently, $\mathbf{v}^t(-\lambda\mathbf{v}) = \lambda\mathbf{v}^t\mathbf{v}$. Hence, $2\lambda\mathbf{v}^t\mathbf{v} = 0$, so that by part (a),

$$2\lambda(v_1^2 + \ldots + v_n^2) = 0, \quad \text{that is } \lambda = 0 \text{ or } \mathbf{v} = \mathbf{0}.$$

$$P = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix},$$

so that the equation is transformed to

$$(\mathbf{x}')^t D\mathbf{x}' + \mathbf{b}^t P\mathbf{x}' + f = 0, \quad \text{that is} \quad 20(x')^2 + 10(y')^2 - \sqrt{10}x' + \sqrt{10}y' - 12 = 0.$$

5. The transformed quadratic equation is $\frac{(x')^2}{2} - \frac{(y')^2}{4} = 1.$

6. The transformed quadratic equation is $\frac{(y')^2}{2} - \frac{(x')^2}{2} = 1.$

7. a. $[x \quad y] \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 16 = 0$ **b.** The action of the matrix

$$P = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

on a vector is a counter clockwise rotation of 45°. Then $P \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} P^t = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}$, so the quadratic equation that describes the original conic rotated 45° is

$$[\,xy\,] \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 16 = 0, \text{ that is } 10x^2 - 12xy + 10y^2 - 16 = 0.$$

8. a. $[x \quad y] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 1 = 0$ **b.** The action of the matrix

$$P = \begin{bmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

on a vector is a clockwise rotation of 30°. Then $P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^t = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$, so the quadratic equation that describes the original conic rotated 30° is

$$[x\ y] \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 1 = 0, \text{ that is } \frac{1}{2}x^2 - \sqrt{3}xy - \frac{1}{2}y^2 - 1 = 0.$$

9. a. $7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0$ **b.** $7(x-3)^2 + 6\sqrt{3}(x-3)(y-2) + 13(y-2)^2 - 16 = 0$

10. a. $\frac{3}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{1}{4}y^2 + \frac{1}{2}x - \frac{\sqrt{3}}{2}y = 0$ **b.** $\frac{3}{4}(x-2)^2 + \frac{\sqrt{3}}{2}(x-2)(y-1)y + \frac{1}{4}(y-1)^2 + \frac{1}{2}(x-2) + \frac{\sqrt{3}}{2}(y-1) = 0$

Exercise Set 6.8

1. The singular values of the matrix are $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}$, where λ_1 and λ_2 are the eigenvalues of A^tA. Then $A^tA = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$, so $\sigma_1 = \sqrt{10}$ and $\sigma_2 = 0$.

2. The singular values of the matrix are $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}$, where λ_1 and λ_2 are the eigenvalues of A^tA. Then $A^tA = \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$, so $\sigma_1 = 2\sqrt{2}$ and $\sigma_2 = \sqrt{2}$.

6. a. Since $\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$ and B is orthonormal so $\langle \mathbf{v}, \mathbf{v_i}\rangle = c_i$, then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}, \mathbf{v_1}\rangle \\ \langle \mathbf{v}, \mathbf{v_2}\rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{v_n}\rangle \end{bmatrix}$.

b. $\text{proj}_{\mathbf{v_i}}\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{v_i}\rangle}{\langle \mathbf{v_i}, \mathbf{v_i}\rangle}\mathbf{v_i} = \langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i} = c_i\mathbf{v_i}$ **c.** The coordinates are given by $c_1 = \langle \mathbf{v}, \mathbf{v_1}\rangle = 1, c_2 = \langle \mathbf{v}, \mathbf{v_2}\rangle = \frac{2}{\sqrt{6}}$, and $c_3 = \langle \mathbf{v}, \mathbf{v_3}\rangle = \frac{1}{\sqrt{6}}\left(\frac{1}{2} + \frac{2}{\sqrt{3}}\right)$.

7. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be an orthonormal basis and $[\mathbf{v}]_B = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix}$. Then there are scalars $c_1, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$. Using the properties of an inner product and the fact that the vectors are orthonormal,

$$\begin{aligned}
\|\ \mathbf{v}\ \| &= \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\langle c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}, c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}\rangle} \\
&= \sqrt{\langle c_1\mathbf{v_1}, c_1\mathbf{v_1}\rangle + \cdots + \langle c_n\mathbf{v_n}, c_n\mathbf{v_n}\rangle} = \sqrt{c_1^2\langle \mathbf{v_1}, \mathbf{v_1}\rangle + \cdots + c_n^2\langle \mathbf{v_n}, \mathbf{v_n}\rangle} \\
&= \sqrt{c_1^2 + \cdots + c_n^2}.
\end{aligned}$$

If the basis is orthogonal, then $\|\ \mathbf{v}\ \| = \sqrt{c_1^2\langle \mathbf{v_1}, \mathbf{v_1}\rangle + \cdots + c_n^2\langle \mathbf{v_n}, \mathbf{v_n}\rangle}$.

8. Let $\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_m\mathbf{v_m}$. Consider

$$\begin{aligned}
0 \le \left\|\mathbf{v} - \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}\right\|^2 &= \left\langle \mathbf{v} - \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}, \mathbf{v} - \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}\right\rangle \\
&= \langle \mathbf{v}, \mathbf{v}\rangle - 2\left\langle \mathbf{v}, \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}\right\rangle + \left\langle \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}, \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle\,\mathbf{v_i}\right\rangle \\
&= ||\mathbf{v}||^2 - 2\sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle^2 + \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle^2 \\
&= ||\mathbf{v}||^2 - \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle^2,
\end{aligned}$$

so

$$||\mathbf{v}||^2 \ge \sum_{i=1}^{m}\langle \mathbf{v}, \mathbf{v_i}\rangle^2.$$

9. a. Since $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ are linearly independent, so $B = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for $\mathbf{col}(A)$.

b. An orthogonal basis is $B_1 = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

27. Let $x \in (A^c)^c$. Then x is in the complement of A^c, that is, $x \in A$, so $(A^c)^c \subseteq A$. If $x \in A$, then x is not in A^c, that is, $x \in (A^c)^c$, so $A \subseteq (A^c)^c$. Therefore, $A = (A^c)^c$.

28. Since an element of the universal set is either in the set A or in the complement A^c, then the universal set is $A \cup A^c$.

29. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B$ and $x \in A$. Hence $x \in B \cap A$. Similarly, we can show that if $x \in B \cap A$, then $x \in A \cap B$.

30. An element $x \in A \cup B$ if and only if

$$x \in A \text{ or } x \in B \Leftrightarrow x \in B \text{ or } x \in A \Leftrightarrow x \in B \cup A$$

and hence, $A \cup B = B \cup A$.

31. Let $x \in (A \cap B) \cap C$. Then $(x \in A$ and $x \in B)$ and $x \in C$. So $x \in A$ and $(x \in B$ and $x \in C)$, and hence, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Similarly, we can show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

32. An element $x \in (A \cup B) \cup C$ if and only if

$$(x \in A \text{ or } x \in B) \text{ or } x \in C \Leftrightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \Leftrightarrow x \in A \cup (B \cup C)$$

and hence, $(A \cup B) \cup C = (A \cup (B \cup C)$.

33. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in (B \cap C)$, so $x \in A$ or $(x \in B$ and $x \in C)$. Hence, $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$. Therefore, $x \in (A \cup B) \cap (A \cup C)$, so we have that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Similarly, we can show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

34. Suppose $x \in A \backslash (B \cap C)$. Then $(x \in A)$ and $x \notin B \cap C$, so $(x \in A)$ and $(x \notin B$ or $x \notin C)$ and hence, $x \in (A \backslash B)$ or $x \in (A \backslash C)$. Therefore, $A \backslash (B \cap C) \subseteq (A \backslash B) \cup (A \backslash C)$. Similarly, we can show that $(A \backslash B) \cup (A \backslash C) \subseteq A \backslash (B \cap C)$.

35. Let $x \in A \backslash B$. Then $(x \in A)$ and $x \notin B$, so $(x \in A)$ and $x \in B^c$. Hence, $A \backslash B \subseteq A \cap B^c$. Similarly, if $x \in A \cap B^c$, then $(x \in A)$ and $(x \notin B)$, so $x \in A \backslash B$. Hence $A \cap B^c \subseteq A \backslash B$.

36. We have that $(A \cup B) \cap A^c = (A \cap A^c) \cup (B \cap A^c) = \phi \cup (B \cap A^c) = B \backslash A$.

37. Let $x \in (A \cup B) \backslash (A \cap B)$. Then $(x \in A$ or $x \in B)$ and $(x \notin (A \cap B))$, that is, $(x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$. Since an element can not be both in a set and not in a set, we have that $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$, so $(A \cup B) \backslash (A \cap B) \subseteq (A \backslash B) \cup (B \backslash A)$. Similarly, we can show that $(A \backslash B) \cup (B \backslash A) \subseteq (A \cup B) \backslash (A \cap B)$.

38. We have that $A \backslash (A \backslash B) = A \backslash (A \cap B^c) = (A \backslash A) \cup (A \backslash B^c) = \phi \cup (A \cap B) = A \cap B$.

39. Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $(y \in B$ and $y \in C)$. So $(x, y) \in A \times B$ and $(x, y) \in A \times C$, and hence, $(x, y) \in (A \times B) \times (A \times C)$. Therefore, $A \times (B \cap C) \subseteq (A \times B) \times (A \times C)$. Similarly, $(A \times B) \times (A \times C) \subseteq A \times (B \cap C)$.

40. We have that

$$\begin{aligned}
(A \backslash B) \cup (B \backslash A) &= (A \cap B^c) \cup (B \cap A^c) = [(A \cap B^c) \cup B] \cap [(A \cap B^c) \cup A^c] \\
&= [(A \cup B) \cap (B^c \cup B)] \cap [(A \cup A^c) \cap (B^c \cup A^c)] \\
&= (A \cup B) \cap (B^c \cup A^c) = [(A \cup B) \cap B^c] \cup [(A \cup B) \cap A^c] \\
&= [(A \cap B^c) \cup (B \cap B^c)] \cup [(A \cap A^c) \cup (B \cap A^c)] \\
&= (A \backslash B) \cup (B \backslash A).
\end{aligned}$$

Exercise Set A.2

1. Since for each first coordinate there is a unique second coordinate, then f is a function.

2. Since $f(1) = -2 = f(4)$, then f is not one-to-one.

12. Notice that consecutive terms in the sum always differ by 4. Then

$$\begin{aligned} 1+5+9+\cdots+(4n-3) &= 1+(1+4)+(1+2\cdot 4)+\cdots+(1+(n-1)\cdot 4) \\ &= n+4(1+2+3+\cdots+(n-1)) \\ &= n+4\left(\frac{(n-1)n}{2}\right) = n+2(n-1)n = 2n^2-n. \end{aligned}$$

13. The base case $n=5$ holds since $32 = 2^5 > 25 = 5^2$. The inductive hypothesis is $2^n > n^2$ holds for the natural number n. Consider $2^{n+1} = 2(2^n)$, so that by the inductive hypothesis $2^{n+1} = 2(2^n) > 2n^2$. But since $2n^2-(n+1)^2 = n^2-2n-1 = (n-1)^2-2 > 0$, for all $n \geq 5$, we have $2^{n+1} > (n+1)^2$.

14.

- Base case: $n=3: 3^2 > 2(3)+1$
- Inductive hypothesis: Assume $n^2 > 2n+1$.

  Using the inductive hypothesis $(n+1)^2 = n^2+2n+1 > (2n+1)+(2n+1) = 4n+2 > 2n+3 = 2(n+1)+1$

15. The base case $n=1$ holds since $1^2+1=2$, which is divisible by 2. The inductive hypothesis is n^2+n is divisible by 2. Consider $(n+1)^2+(n+1) = n^2+n+2n+2$. By the inductive hypothesis, n^2+n is divisible by 2, so since both terms on the right are divisible by 2, then $(n+1)^2+(n+1)$ is divisible by 2. Alternatively, observe that $n^2+n = n(n+1)$, which is the product of consecutive integers and is therefore even.

16.

- Base case: $n=1$: Since $(x-y) = 1\cdot(x-y)$, then $x-y$ is divisible by $x-y$.
- Inductive hypothesis: Assume x^n-y^n is divisible by $x-y$.

  Consider

$$\begin{aligned} x^{n+1}-y^{n+1} &= x\cdot x^n - y\cdot y^n = x\cdot x^n - y\cdot x^n + y\cdot x^n - y\cdot y^n \\ &= x^n(x-y)+y(x^n-y^n). \end{aligned}$$

  Since $x-y$ divides both terms on the right, then $x-y$ divides $x^{n+1}-y^{n+1}$.

17. For the base case $n=1$, we have that the left hand side of the summation is 1 and the right hand side is $\frac{r-1}{r-1} = 1$, so the base case holds. For the inductive hypothesis assume the summation formula holds for the natural number n. Next consider

$$1+r+r^2+\cdots+r^{n-1}+r^n = \frac{r^n-1}{r-1}+r^n = \frac{r^n-1+r^n(r-1)}{r-1} = \frac{r^{n+1}-1}{r-1}.$$

Hence, the summation formula holds for all natural numbers n.

18. a., b.

n	f_n	$f_1+f_2+\cdots+f_n$
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33

The pattern suggests that $f_1+f_2+\cdots+f_n = f_{n+2}-1$.

c.

- Base case: $n=1: f_1 = f_3-1$