**Theorem.** If there are strictly positive constants \(c_1\) and \(c_2\) such that
\[
c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)
\]
for all \(x, y \in X\), then \(d_1\) and \(d_2\) are topologically equivalent metrics on \(X\).

**Proof.**

Let \(U\) be open in \((X, d_1)\).

Then for \(a \in X\) there exists \(\epsilon > 0\) such that
\[
\{x \in X; d_1(x, a) < \epsilon\} \subset U.
\]

But then
\[
\{x \in X; d_2(x, a) < c_1 \epsilon\} \subset \{x \in X; d_1(x, a) < \epsilon\} \subset U
\]
and \(U\) is open in \((X, d_2)\).

Similarly, if \(U\) is open in \((X, d_2)\) it is open in \((X, d_1)\), and the metrics are topologically equivalent.

This criterion is sufficient but not necessary.

Let \(X = \mathbb{R}\), and consider the metrics
\[
d_1(x, y) = |x - y|
\]
\[
d_2(x, y) = \frac{|x - y|}{1 + |x - y|}
\]
Obviously \(d_2(x, y) \leq d_1(x, y)\) for all \(x, y\), but since \(d_1(x, y) = \left(1 + |x - y|\right)d_2(x, y)\) there is no strictly positive constant \(c\) such that \(d_2(x, y) \geq cd_1(x, y)\).

On the other hand
\[
\{d_1(x, a) < \epsilon\} \subset \{d_2(x, a) < \epsilon\}
\]
so that if \(U\) is open in \((\mathbb{R}, d_2)\) it is open in \((\mathbb{R}, d_1)\), while if \(|x - a| < \epsilon\),
\[
d_1(x, a) = \left(1 + |x - a|\right)d_2(x, a)
\]

Let \(\epsilon_1 = \epsilon/(1 + \epsilon)\).

Then \(\{d_1(x, a) < \epsilon_1\} \subset \{d_2(x, a) < \epsilon\}\), so that if \(U\) is open in \((\mathbb{R}, d_1)\) it is open in \((\mathbb{R}, d_2)\).

For example, in \(\mathbb{R}^2\),
\[
\max(|x_1 - x_2|, |y_1 - y_2|)
\]
\[
\leq |x_1 - x_2| + |y_1 - y_2|
\]
\[
\leq 2 \max(|x_1 - x_2|, |y_1 - y_2|)
\]
so that the taxi-cab and sup metrics are equivalent.

In fact all the metrics generated by the norms \(||x||_p\) are topologically equivalent on \(\mathbb{R}^n\) for finite \(n\).