2) If \( Y = l \), then the conditional distribution of \( X \mid (Y = l) \) is Binomial \( (n - l, \frac{p}{1 - \theta}) \).

**Proof.**

\[
P(X = k \mid Y = l) = \frac{P(X = k, Y = l)}{P(Y = l)} = \frac{n!}{k!(n-k)!}p^k \theta^l (1 - p - \theta)^{n-k-l}
\]

\[
= \binom{n-l}{k} \left( \frac{p}{1 - \theta} \right)^k \left( 1 - \frac{p}{1 - \theta} \right)^{n-l-k}
\]

for \( x = 0, 1, \ldots, (n - y) \). Hence \( X \mid Y = y \sim \text{Binomial} \ (n - y, \frac{p}{1 - \theta}) \). \( \square \)

This is intuitively obvious. Consider those trials for which “failure” (or 0) did not occur. There are \( (n - l) \) such trials, for each of which the probability that 1 occurs is actually the conditional probability of 1 given that 0 has not occurred, i.e. \( \frac{p}{1 - \theta} \). So you have the standard binomial set-up.

3) We shall now use the results on conditional distributions (Notes 5) and the above properties to find \( \text{Cov}(X, Y) \) and the coefficient of correlation \( \rho(X, Y) \).

We proved that \( E[XY] = E[YE[X \mid Y]] \) (see the last page of Notes 5). According to property 2), \( E[X \mid Y = l] = (n - l) \frac{p}{1 - \theta} \) and thus \( E[X \mid Y] = (n - Y) \frac{p}{1 - \theta} \). Hence

\[
E[XY] = E\left[ Y \times (n - Y) \frac{p}{1 - \theta} \right] = \frac{p}{1 - \theta} E(nY - Y^2) = \frac{p}{1 - \theta} \left( n^2 \theta - n \theta (1 - \theta) + n^2 \theta^2 \right)
\]

\[
= \frac{p}{1 - \theta} [n(n - 1) \theta (1 - \theta)] = n(n - 1) \theta p
\]

Therefore \( \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = n(n - 1) \theta p n^2 p \theta = -np \theta \) and hence

\[
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{-np \theta}{\sqrt{n^2 p (1 - p) \theta (1 - \theta)}} = -\left( \frac{p \theta}{(1 - p)(1 - \theta)} \right)^{\frac{1}{2}}
\]

Note that if \( p + \theta = 1 \) then \( Y = n - X \) and there is an exact linear relation between \( Y \) and \( X \). In this case it is easily seen that \( \rho(X, Y) = -1 \).

**Definition of the multinomial distribution**

Now suppose that there are \( k \) outcomes possible at each of the \( n \) independent trials. Denote the outcomes \( A_1, A_2, \ldots, A_k \) and the corresponding probabilities \( p_1, \ldots, p_k \) where \( \sum_{j=1}^{k} p_j = 1 \). Let \( X_j \) count the number of times \( A_j \) occurs. Then

\[
P(X_1 = x_1, \ldots, X_{k-1} = x_{k-1}) = \frac{n!}{x_1!x_2! \ldots x_{k-1}!(n - \sum_{j=1}^{k-1} x_j)!} p_1^{x_1} p_2^{x_2} \ldots p_{k-1}^{x_{k-1}} p_k^{n - \sum_{j=1}^{k-1} x_j}
\]

where \( x_1, x_2, \ldots, x_{k-1} \) are non-negative integers with \( \sum_{j=1}^{k-1} x_j \leq n \).