1.5.1 Example 5

We can show that

\[ \int_a^x (t + 1) \, dt = \frac{1}{2}(x^2 - a^2) + (x - a) \]

This means that \( A(x) = \int_a^x (t + 1) \, dt \). Differentiating \( A(x) \) we get,

\[ A'(x) = \frac{1}{2}(2)(x) + 1 = x + 1 = f(x) \]

1.5.2 Example 6

Let \( f(x) = 3x^2 \). We observe that if \( F(x) = x^3 \) then \( F'(x) = 3x^2 = f(x) \). Part (2) of the fundamental theorem of Calculus states,

\[ \int_a^b 3x^2 \, dx = F(b) - F(a) = b^3 - a^3 \]

1.6 Proof of the fundamental theorem of Calculus

If \( f \) is a continuous function and \( F \) is any function such that \( F'(x) = f(x) \), then \( F \) is said to be an antiderivative of \( f \). \( F \) is also known as the primitive integral or the indefinite integral.

1.6.1 Part 1

\[ A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h} \]

\[ A(x + h) - A(x) = \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_a^x f(t) \, dt + \int_x^{x+h} f(t) \, dt \]

From the mean value theorem we have,

\[ \frac{1}{x+h-x} \int_x^{x+h} f(t) \, dt = f(c) = \frac{1}{h} \int_x^{x+h} f(t) \, dt \implies hf(c) = \int_x^{x+h} f(t) \, dt \]

For some \( c = c_h \in (x, x+h) \), meaning \( x < c_h < x+h \).

\[ A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \lim_{h \to 0} f(c) = \lim_{h \to 0} f(c_h), \]

With \( x < c_h < x+h \). By the sandwich theorem as \( h \to 0, c_h \to x \), so \( A'(x) = f(x) \).

1.6.2 Part 2

If \( F'(x) = f(x) \) then, \( \int_a^b f(x) \, dx = F(b) - F(a) \). From part (1), \( A(x) = \int_a^x f(t) \, dt \) satisfies \( A'(x) = f(x) \).

Introducing the equation \( G(x) = F(x) - A(x) \implies G'(x) = F'(x) - A'(x) \implies G(x) = f(x) - f(x) = 0 \). So \( G \) is a constant, \( G(x) = c \). So \( F(x) = A(x) + G(x) \implies F(x) = A(x) + c \). So,

\[ F(x) = \int_a^x f(t) \, dt + c \]

From the original statement,

\[ F(b) - F(a) = (\int_a^b f(t) \, dt + c) - (\int_a^a f(t) \, dt + c) = \int_a^b f(t) \, dt \]

as \( \int_a^a f(t) \, dt = 0 \) from integral property (5).

To conclude, the fundamental theorem of Calculus says (1) differentiation is the inverse of integration. (2) integration is the inverse of differentiation.