There are also some surprising ways to use the theorem. For example, let \( n \in \mathbb{Z}^+ \), and let \( 0 \leq m \leq n \). For any positive integer \( k \), \( n^k \) can be expressed as a sum of powers of \( m \) and \( n - m \). To see this, simply note that, by the Binomial Theorem,

\[
n^k = \sum_{j=0}^{k} \binom{k}{j} m^j (n - m)^{k-j}.
\]

For an example, \( 5^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k} \).

Here are some additional examples of combinatorial proof.

**Example 4**: A nameless algebraic identity states that
\[
\binom{2n}{2} = 2 \binom{n}{2} + n^2.
\]

Here is a combinatorial solution. Its use of color is just one of several ways to differentiate the elements of the two subsets introduced to drive the proof.

**Proof**: The expression on the left-hand side is the number of 2-subsets of a 2\( n \)-set. Let \( A \) be a 2\( n \)-set, and suppose that \( A \) contains \( n \) red elements and \( n \) blue elements. We now choose all the possible 2-subsets, by counting all the choices: all the 2-subsets that have exactly 2 red elements, all the 2-subsets that have exactly 2 blue elements, and all the 2-subsets that have exactly one red element \( A \) and one blue element. There are \( \binom{n}{2} \) red 2-subsets, \( \binom{n}{2} \) blue 2-subsets, and \( \binom{n}{2} \) \( \binom{n}{1} \) 2-subsets containing one red and one blue elements. By the Sum Rule, the number of 2-subsets of \( A \) is \( 2 \binom{n}{2} + n^2 \). \( \square \)

**Example 5**: Here is a variation on the theme. Suppose we want to prove the identity,
\[
\binom{2n}{3} = 2 \binom{n}{3} + 2n \binom{n}{2}.
\]

The same technique used in the preceding problem leads to the following argument.

**Proof**: The expression on the left counts the number of 3-subsets of a 2\( n \)-set. Let \( A \) be a 2\( n \)-set containing \( n \) red and \( n \) blue elements. There are \( \binom{n}{3} \) red 3-subsets, \( \binom{n}{3} \) blue 3-subsets, \( \binom{n}{2} \binom{n}{1} \) 3-subsets with two red elements and one blue, and \( \binom{n}{2} \binom{n}{1} \) 3-subsets with two blue elements and one red. Simplifying, we see that the number of 3-subsets of \( A \) is given by \( \binom{n}{3} + \binom{n}{2} \binom{n}{1} + \binom{n}{2} \binom{n}{1} = 2 \binom{n}{3} + 2n \binom{n}{2} \). The result follows. \( \square \)

**Example 6**: Here’s another, asking for a proof of the identity
\[
\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}.
\]