• Secondary markets can be exchanges (all offers are considered simultaneously and the price is set by supply and demand) or over the counter (OTC) markets (traders seek counterparties in a less organized manner);

• Terminology:
  - Broker: middleman who connects sellers and buyers;
  - Dealer: trades on his own account;
  - Market maker: offers buying and selling prices to other market participants;
  - Specialist: matches buying and selling offers according to auction mechanism but may also trade on his own account;

3. Double auctions

• Most exchanges are double auctions where both buyers and sellers bid for securities and the most generous offers are selected to participate in trades;

• Exchanges can be call auctions or continuous auctions:
  (1) In a call auction, the market is cleared once at the end of a bidding period by matching the maximum amount of buying and selling offers;
  (2) In a continuous auction, a new offer is matched immediately with existing offers if possible, otherwise, the new offer is added to the limit order book;

• Call auctions:
  - Selling offers: a piecewise constant non-decreasing function \( x \rightarrow s(x) \), the supply curve;
  - Buying offers: a piecewise constant non-increasing function \( x \rightarrow d(x) \), the demand curve;
  - The market is cleared by matching maximum number of trades: \( \bar{x} = \sup \{ x \mid s(x) \leq d(x) \} \), and the interval \( \left[ \lim_{x \downarrow \bar{x}} d(x), \lim_{x \uparrow \bar{x}} s(x) \right] \) consists of the market clearing prices;
  - Market clearing can be interpreted as finding the social optimum: \( S(x) = \int_0^x s(z)dz \) and \( D(x) = \int_0^x d(z)dz \) may be interpreted as the cost of producing \( x \) units, and the value of consuming \( x \) units;
  - Market is cleared by minimizing the difference \( S(x) - D(x) \);

• Convex analysis:
  - For a real – valued function \( f \) on an interval \( I \) allowing are equivalent:
    (a) \( f \) is convex,
    (b) There is a non – decreasing function \( \emptyset : I \rightarrow R \) such that \( f(x) = f(\bar{x}) + \int_\emptyset^x \emptyset(z)dz \) for all \( x, \bar{x} \in I \)
    (c) \( f \) is differentiable on \( I \) except a set on a countable set, its derivative \( f' \) is non – decreasing and \( f' = f' = \emptyset \);

  Proof of \( (b) \Rightarrow (a) \): Let \( x_i \in I \) such that \( x_1 < x_2 \) and \( \alpha_i > 0 \) such that \( \alpha_1 + \alpha_2 = 1 \).

  With \( x = \alpha_1 x_1 + \alpha_2 x_2 \), we have: \( f(x) - f(x_1) = \int_{x_1}^x \emptyset(x)dz \leq \int_{x_1}^x \emptyset(x)dz = \emptyset(x)(x - x_1) \)

  And \( f(x_2) - f(x) = \int_{x_1}^{x_2} \emptyset(x)dz \geq \int_{x_1}^{x_2} \emptyset(x)dz = \emptyset(x)(x_2 - x) \)

  Multiply the inequalities by \( \alpha_1 \) and \( - \alpha_2 \) and add up: \( f(x) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) \), thus \( f \) is convex.

  - Note that \( \emptyset \) in \( (b) \) is not unique: any \( \emptyset \) with \( f' \leq \emptyset \leq f' \) will do;
  - Subgradient: A \( v \in R \) is a subgradient at \( \bar{x} \) if \( v \in [f'(\bar{x}), f'(\bar{x})] \); and the set of subgradients of \( f \) at \( \bar{x} \) is known as the subdifferential of \( f \) at \( \bar{x} \) and is denoted by \( \partial f(\bar{x}) \); Note that an \( \bar{x} \in I \) minimizes \( f \) over \( I \) if and only if \( 0 \in \partial f(\bar{x}) \);
  - Back to market clearing, an \( \bar{x} \) minimizes \( S(x) - D(x) \) iff \( 0 \in \partial(S - D)(\bar{x}) \), which means:
    \[ 0 \in [s_{-} - d_{-}(\bar{x}), s_{+}(\bar{x}) - d_{+}(\bar{x})] \]

• Limit order book (LOB): where the offers remaining after market clearing are recorded; LOB gives the marginal prices for buying or selling a given quantity at the best available prices;

• Flatter the curve \( s \), more liquid the market;

• Continuous auction: market is cleared very frequently;

• Limit sell orders above the best ask-price and limit buy orders below the best bid-price increase liquidity; A market order is an order to buy/sell a given amount at the best available prices, which reduce liquidity;

• Price per share \( S(x) = \frac{s(x)}{x} \), the convexity of \( S \) gives us: if \( x_1 < x_2 \)