is not in the subset since the third entry does not satisfy the formula \(2a - b + c = 1\).

Therefore, the subset of all vectors of the form \[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\]
where \(2a - b + c = 1\) is not a subspace of \(\mathbb{R}^3\).

6. Consider the circle in the \(xy\)-plane centered at the origin whose equation is \(x^2 + y^2 = 1\). Let \(W\) be the set of all vectors whose tail is at the origin and whose head is a point inside or on the circle. Is \(W\) a subspace of \(\mathbb{R}^2\)? Explain.

\[
\begin{array}{c}
\text{Solution} \\
\text{Note that the radius of the circle is 1. Thus, any vector }
\begin{pmatrix}
x \\
y
\end{pmatrix}
\text{whose head lies outside the circle is not a vector in } W. \\
\text{Take a vector } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ in } W \text{ and let a scalar } r \text{ to be 2.} \\
\text{Hence, the scalar multiple } \\
r \mathbf{u} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}
\text{is not in } W. \text{ Therefore, } W \text{ is not a subspace.}
\end{array}
\]

7. Determine whether the subsets of all matrices of the following forms of \(M_{23}\) are subspaces.

(a) \[
\begin{pmatrix}
a & b & c \\
d & 0 & 0
\end{pmatrix}, \text{ where } b = a + c
\]

(b) \[
\begin{pmatrix}
a & b & c \\
d & 0 & 0
\end{pmatrix}, \text{ where } c < 0
\]

(c) \[
\begin{pmatrix}
a & b & c \\
d & e & f
\end{pmatrix}, \text{ where } a = -3c \text{ and } f = 3e + d
\]

\[
\begin{array}{c}
\text{Solution}
\end{array}
\]
The corresponding linear system is
\[\begin{align*}
a_1 + a_2 &= a \\
-2a_1 + 2a_2 &= b \\
3a_1 - a_2 &= c \\
3a_3 &= d
\end{align*}\]

Form the augmented matrix and row reduce it.
\[
\begin{bmatrix}
1 & 1 & 0 & a \\
-2 & 2 & 0 & b \\
3 & -1 & 0 & c \\
0 & 0 & 3 & d
\end{bmatrix}
\]

The system is consistent only when \( a + b - a - c + 3a = 0 \). Therefore, the vectors do not span \( \mathbb{R}_4 \).

(c) \([1 1 0 0], [1 2 -1 1], [0 1 1], [2 1 2 1]\)

Solution

Let \([a b c d]\) be an arbitrary vector in \( \mathbb{R}_4 \). So we have
\[
a_1[1 1 0 0] + a_2[1 2 -1 1] + a_3[0 0 1 1] + a_4[2 1 2 1] = [a b c d].
\]

The corresponding linear system is
\[\begin{align*}
a_1 + a_2 + 2a_4 &= a \\
\quad a_1 + 2a_2 + a_4 &= b \\
-\quad a_2 + a_3 + 2a_4 &= c \\
\quad a_2 + a_3 + a_4 &= d
\end{align*}\]

Form the augmented matrix and row reduce it.
\[
\begin{bmatrix}
1 & 1 & 0 & 2 & a \\
1 & 2 & 0 & 1 & b \\
0 & -1 & 1 & 2 & c \\
0 & 1 & 1 & 1 & d
\end{bmatrix}
\]
Solution

\[
\begin{bmatrix}
3 & 1 & 4 & 1 \\
1 & 2 & 3 & 1 \\
1 & 1 & 2 & 0 \\
1 & 1 & 2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 1 & 5 & -3 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

The system has many solutions. Therefore, \( S \) is not linearly independent.

14. Which of the given vectors in \( \mathbb{R}^3 \) are linearly dependent? For those which are, express one vector as a linear combination of the rest.

(a) \([2 \ -1 \ 0],[0 \ 3 \ 2],[2 \ 4 \ 3],[3 \ 6 \ 6]\)

Form the equations:

\[a_1[2 \ -1 \ 0]+a_2[0 \ 3 \ 2]+a_3[2 \ 4 \ 3]+a_4[3 \ 6 \ 6]=[0 \ 0 \ 0].\]

The corresponding linear system is

\[
\begin{align*}
2a_1 + 2a_3 + 3a_4 &= 0 \\
-a_1 + 3a_2 + 4a_3 + 6a_4 &= 0 \\
2a_2 + 3a_3 + 6a_4 &= 0
\end{align*}
\]

Form the augmented matrix and perform row reduction:

\[
\begin{bmatrix}
2 & 0 & 2 & 3 & 0 \\
-1 & 3 & 4 & 6 & 0 \\
0 & 2 & 3 & 6 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -4 & -6 & 0 \\
0 & 2 & 3 & 6 & 0 \\
0 & 6 & 10 & 15 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 3 & 0 \\
0 & 1 & 5 & 3 & 0 \\
0 & 0 & 1 & -3 & 0
\end{bmatrix}
\]

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Solution
Form the equation:

\[ a_1(t^2 + 1) + a_2(t - 2) + a_3(t + 3) = 0t^2 + 0t + 0, \]
\[ a_1t^2 + (a_2 + a_3)t + (a_1 - 2a_2 + 3a_3) = 0t^2 + 0t + 0. \]

Equating coefficients of like powers of \( t \), we obtain the system

\[ a_1 = 0, \quad a_2 + a_3 = 0, \quad a_1 - 2a_2 + 3a_3 = 0. \]

Next, row reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & -2 & 3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 5 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

The solution is \( a_1 = a_2 = a_3 = 0 \), therefore the vectors are linearly independent.

(b) \( 2t^2 + t, t^2 + 3, t \)

Solution
Form the equation:

\[ a_1(2t^2 + t) + a_2(t^2 + 3) + a_3t = 0t^2 + 0t + 0, \]
\[ (2a_1 + a_2)t^2 + (a_1 + a_3)t + 3a_2 = 0t^2 + 0t + 0. \]

Equating coefficients of like powers of \( t \), we obtain the system

\[ 2a_1 + a_2 = 0, \quad a_1 + a_3 = 0, \quad 3a_2 = 0. \]

Next, row reduce the augmented matrix of the system:

\[
\begin{bmatrix}
2 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 3 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
0 & 3 & 0 & 0
\end{bmatrix}
\]
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} - 3R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$
$$\frac{1}{2}R_3 \rightarrow R_3$$

The solution is $a_1 = a_2 = a_3 = 0$. Therefore the vectors are linearly independent.

(c) $2t^2 + t + 1, 3t^2 + t - 5, t + 13$

**Solution**

Form the equation:

$$a_1(2t^2 + t + 1) + a_2(3t^2 + t - 5) + a_3(t + 13) = 0t^2 + 0t + 0,$$
$$2a_1 + 3a_2 = 0$$
$$a_1 + a_2 + a_3 = 0.$$

Equating coefficients of like powers of $t$, we obtain the system:

$$2a_1 + 3a_2 = 0$$
$$a_1 + a_2 + a_3 = 0.$$

Next, row reduce the augmented matrix of the system:

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & -5 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -6 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is $a_3 = r, a_2 = 2r, a_1 = -3r$. Therefore the vectors are linearly dependent. Take $r = 1$, we have

$$a_1(2t^2 + t + 1) + a_2(3t^2 + t - 5) + a_3(t + 13) = 0t^2 + 0t + 0,$$
$$-3(2t^2 + t + 1) + 2(3t^2 + t - 5) + (t + 13) = 0t^2 + 0t + 0,$$
$$(t + 13) = 3(2t^2 + t + 1) - 2(3t^2 + t - 5).$$

16. Which of the given vectors in $\mathbb{R}^3$ are linearly dependent? For those which are, express one vector as a linear combination of the rest.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$
of $A$ is $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 3 \end{bmatrix}$.

32. Find a basis for the column space of $A$ consisting of vectors that (a) are not necessarily column vectors of $A$; and (b) are column vectors of $A$.

$$A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}.$$ 

Solution

We could perform column reduction to transform the matrix into rref. However, here we transpose matrix $A$ and perform row reduction.

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & -1 & 2 & 1 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix} \xrightarrow{2R_1+R_2-R_3} \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-7R_1+R_4-R_3} \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{3R_1+R_3-R_4} \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2-R_4} \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

(a) The basis for the row space of $A$ consisting of vectors that are not necessarily row vectors of $A$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}$. Therefore the basis for column space is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

(b) The basis for column space of $A$ consisting of vectors that are column vectors of $A$ is

$$\begin{bmatrix} 1 & -2 & 7 \\ 1 & -1 & 4 \\ 3 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}.$$ 

33. Find the row and column ranks of the given matrices.