Part 1 (Interest Theory)

Other Formulas and Principles

- 99% of the time, my mistake is a mistake of retranscription in one of the formulas, a mistake in what to calculate or a mistake in not using the good force of interest.

NB. - Don’t waste your time recalculating with the calculator.

- Calculator: Format AOS and 8 decimals

- Outlay = payment

- 60 through 72: [60; 72]

- One payment every other year = one payment every 2 years

- She borrows X that she intends to repay 5 years later at $t = 5, X(1+i)^5$ not $X$

- A is greater than B: $A = 8 + B$ not $A = 8B$

- Write in $M$ if there are 6 digits or more to simplify the writings. Furthermore, don’t take into account decimals if there are 4 digits or more at the left. If not, take only 2 decimals.

- Each payment will be 3% less than the preceding payment: $0.97^{t-1}$ not $(1.03)^{t-1}$

- At the end of 20 years, the total in the two funds is 10,000:

$$\sum \text{funds} = 10,000 \text{ not } \text{fund}_1 = \text{fund}_2 = 10,000$$

- Always put the force of interest in function of payment frequency to have this: $t = 0, 1, 2, \ldots$ For example, $I_t = iB_{t-1}$ with payments are every 2 years with: $t = 0, \frac{1}{2}, \ldots$
Separate funds:  

\[ B_0 f_δ_i(0; 1) \quad B_0 f_δ_i(n - 1; n) \]

|-------|--------------------------------------------|--------> \( \delta_j \) |
| 0     | 1           | ... | n |

\[ B_0 f_δ_i(1; 2) \]

| |-----------------------------|--|--------> \( \delta_j \) |
| 0 | 2           | ... | n |

...  

\[ B_0 f_δ_i(n - 1; n) \]

| |-------------------------------------------|--------> \( \delta_j \) |
| 0 | ... | ... | n |

\[ B_0 f_δ_i(0; 1) f_δ_j(1; 2) \quad \sum_{t=1}^{n-1} B_0 f_δ_i(t - 1; t) f_δ_j(t; t + 1) \]

Side funds:  

|-------|---------------------------|--|--------> \( \delta_k \) |
| 0     | ... | ... | n |

\[ B_n = B_0 + \sum_{t=1}^{n} B_0 f_δ_i(t - 1; t) \]

\[ + \sum_{p=1}^{n} \sum_{t=1}^{p} B_0 f_δ_i(t - 1; t) f_δ_j(t; t + 1) \prod_{t=p+2}^{n} (1 + f_δ_k(t - 1; t)) \]

NB. - It doesn’t change anything that the funds are separated (for the individual separate funds).

- With the force of interest of \( i, j \) and \( k \) for these 3 funds:

\[ B_n = B_0 + n(iB_0) + (ijB_0)(I_s)_{n-1|k}. \]

- Interest reinvested in a different fund with \( R \) level payments (more simple case):

\[ R \quad R \]

| |-----------------------------|--|--------> \( i \) |
| 0 | 1           | ... | n |
b) Lender’s point of view:

\[ L(1 + r)^n = Lis_n \lfloor \frac{i}{x} \rfloor + L \]

**Bond:**

- Without reinvestment:

\[ P(1 + r)^n = \text{coupon} \ast s_n \lfloor \frac{i}{x} \rfloor + C, \]

\( \text{coupon} = Fr \)

- With reinvestment:

\[ P(1 + r)^n = \text{coupon} \ast s_n \lfloor \frac{i}{x} \rfloor + C \]

**NB.** \( r \), the overall yield, has nothing to do with \( r \), the coupon rate.

**Short selling:**

\[ \text{Haircut}(1 + r)^n = FV(S_0) \delta \text{ or } r_e - S_r + \text{Haircut}(1 + i_H)^n \]

- For \( r_A \) and \( r_B \) from \( 0 \) to \( t \) and \( t \) to \( n \)

**Callable bond with A who sells the bond to B at time t with reinvested coupons:**

\[ P_{\text{min}}(0; t) = \text{price of callable bond} = F_r s_t \lfloor \frac{i}{x} \rfloor + C_t, \]

(to calculate \( P_{\text{min}}(0; t) \))

\[ P_{\text{min}}(0; t)(1 + r_A)^t = F_r s_t \lfloor \frac{i}{x} \rfloor + C_t, \]

(to isolate \( C_t \))

where \( P_{\text{min}}(0; t) = \text{price of the callable bond} = F_r a_t \lfloor \frac{1}{x} \rfloor + C_t v^t \)

\[ C_t = \text{call price of the bond} = \text{sold price of the bond} \]
\[ C_t (1 + r_B)^{n-t} = Frs_{n-t} + C_n, \]  

(to find \( r_B \))

where \( C_t \) = sold price of the bond, \( t \)

\( C_n \) = maturity value

NB. –“P” is not necessary equal to \( P_{\text{min}}(0; t) \)

-“r”, the coupon rate, has nothing to do with \( r_A \) and \( r_B \), the overall yields.

-If there was no \( j_A \), it would be \( i_A \) to isolate \( C_t \).

-Because there are no reinvested coupons for B, we use \( r_B \) and not \( j_B \).

- Internal rate of return (IRR) with NPV:

\[ \rightarrow \text{If the investment comes from the company:} \]

\[ \text{NPV} = -\text{Investment} - PV(\text{revenues})_{1RR,j} - PV(\text{expenses})_{1RR,j}, \]  

(to find IRR)

\[ = 0 \]

where \( PV(\text{revenues})_{1RR,j} = \sum_{k=1}^{n} CF_{t_k}(1+j)^{t_n-t_k} \)

\( n \) = number of cash inflows

\( t_k \) = time of the \( k^{\text{th}} \) moment

\( j \) = reinvestment rate

\[ \rightarrow \text{If there is a loan as the investment:} \]

\[ \text{NPV} = L - L_{\text{loan}} a_{n|1RR} - L v_{1RR}^n - PV(\text{expenses})_{1RR} \]
\[ t = t^{th} \text{moment} \]
\[ m = \text{number of times/year the interest applies} \]
\[ B_{t_2} = B_{t_1} (1 + i)^{t_2-t_1} \sum_{j=t_1+1}^{t_2} CF_j (1 + i)^{t_2-j}, \quad t_1 < t_2 \]

NB. -Same thing as bonds at premium.

- Amortization method:
  -the accumulated amount is paid at the end as a lump sum:

\[ \sum I = L (1 + i)^{n} - L \]

- level payments:

\[ \sum I = nR - L, \]
where \( L = Ra_n|i \)

- level principal payments:

\[ \sum I = \sum_{t=1}^{n} f(t-1; t)(L - (t - 1)X), \]
where \( R_t = X + I_t \)
Chap. 7 (Bonds)

- The annual coupon rate is 7% (with semi-annual coupons): \( r^{(2)} = 7\% \)
- $5000 par bond: \( F = 5000 \)
- A par coupon bond = $1 par bond
- A 4% $100 bond with semi-annual coupons: \( r^{(2)} = 4\% \) and \( F = 100 \)
- A $10,000 5-year equity-linked CD: \( F = 10,000 \)
- To purchase a bond at a price of 1700 (for a callable bond): \( P_{\text{min}}(0; t) = 1700 \) not \( P = 1700 \)

NB. - They don’t say it’s a bond price for a callable bond. They just say it’s the price of the bond.

- The bond is sold at a price equal to its value = redeemable at par = callable at par = to call at, “to redeem at” and “to mature at”.
- \( F = C \) and \( r = i \Rightarrow P = F = C \),
  where \( C_t \) is not necessary equal to the maturity value (\( C \) at time \( n \)).
- By default, \( F = C \) if not mentioned (only seen in the spot rate numbers).
- \( P_{\text{min}}(0; t) = \) minimum redemption value = minimum call price = call price
  NB. - It is not necessary equal to \( P \) or \( C_n \).
- Accumulation bond = zero-coupon bond
- Bond with annual coupons of 6.75% at par = to buy a bond at par
  = bond A is priced at par = bond sells at par
  = to buy the bond with no premium or discount:
• ModD = vMacD = $-\frac{p'}{p}$

NB. ModD was developed by drawing a tangent line to the price curve:

• Convexity$_L = \frac{p''_L}{p_L}$ is not necessary equal, smaller or bigger than:

$$\sum_{j=1}^{n} w_j Convexity_{A_j} = \frac{p''_A}{p_A}$$

• Change in the interest rate (in A or L):

$$P_{i\text{ after}} = P_{i\text{ before}} + \Delta P,$$

where $\Delta P \approx -P_{i\text{ before}} \cdot \Delta i \cdot \text{ModD}$ (approximation with 1 term)

$$\Delta i = i_{\text{after}} - i_{\text{before}}$$

• If there is a CF (A or L) at $t = 0$:

\[
\begin{cases}
\text{Defer all the CF (L and A) of on } w_j, \text{ with exact matching} \\
\text{It doesn't change anything with RI and FI} \\
\end{cases}
\]

(to find $w_j$)

• Immunization

- Characteristics:

  ➔ Either MacD or ModD can be used to develop an immunization strategy
  ➔ The yield curve structure is not relevant
  ➔ Matching the PVs is not sufficient when the interest rates change

• To immunize = to exactly match = to exactly (absolutely) match = to match = to produce exact matching = method if not mentioned how to immunize
The yield curve shifts in parallel when the interest rate changes

-if “i” is the same for A and L:

1) $\text{MacD} = \sum_{j=1}^{n} w_j MacD_{A_j}$

2) $P_L = \sum_{j=1}^{n} P_{A_j}$

where $P_{A_j} = w_j P_L$

NB. -Or we can just use the second equation for 2).

-otherwise (conditions for Redington immunization):

1) $P_A = P_L$

2) $P_A' = P_L'$

3) $P_A'' > P_L''$

NB. -$P_A' = P_L'$ comes from $\text{ModD} = \frac{P_A'}{P_A} = \frac{P_L'}{P_L} = \text{ModD}_L$ and $P_A'' > P_L''$

comes from $\text{Convexity}_A = \frac{P_A''}{P_A} > \frac{P_L''}{P_L} = \text{Convexity}_L$.

To validate the RI conditions, we must find $P_{A_1}$ and $P_{A_2}$ and then validate with 3).

3- Full immunization:

-Characteristic:

- It protects against any change in the interest rate

-if “i” is the same for A and L:

1) $\text{MacD} = \sum_{j=1}^{n} w_j MacD_{A_j}$

2) $P_L = \sum_{j=1}^{n} P_{A_j}$ or $P_{A_j} = w_j P_L$
We buy a forward to lock in the purchase price (we want $S_T = 1,025$). He wants to sell an asset at $t = T$.

- **F or $F(S)$:**

  ![Diagram showing payoff vs. $S_T$]

  **Payoff**

  

  $F$

  $S_T$

  **NB.** -Slope = 1 or -1.

- **Forward:**

  \[
  \text{Profit}_T = n \text{payoff} - \text{FV(premium)},
  \]

  where $n$ = number of options

  \[
  \text{payoff} = \begin{cases} 
  S_T - F, & \text{with a long forward} \\
  F - S_T, & \text{with a short forward} 
  \end{cases}
  \]

  $F$ = forward price = observed forward price = current forward price

  \[
  \text{premium} = \begin{cases} 
  \text{PV}(F) - \text{PV}(K) = C_k - P_k, & \text{with a long forward} \\
  \text{PV}(K) - \text{PV}(F) = P_k - C_k, & \text{with a short forward} 
  \end{cases}
  \]

  \[
  = \begin{cases} 
  0, & \text{usually} \\
  \neq 0, & \text{with an off - market forward}
  \end{cases}
  \]

  \[
  \text{PV}(F) = F^p = \begin{cases} 
  S_t - \text{PV(dividends)}, & \text{with discrete dividends} \\
  S_t e^{\delta (T-t)}, & \text{with continuous dividends} \\
  \end{cases}
  \]

  with no dividends

  \[
  = \text{prepaid forward}
  \]

**NB.** –There is one contract.
This strategy locks in the selling price (for example, wheat) between $K_1$ and $K_2$.

NB. -Strategy = collar, but with a stock.

-> zero-cost collar:

a) the put can be at-the-money: $K_1 - S_0 = 0 \Rightarrow K_1 = S_0$

b) the call cannot be at-the-money: $S_0 - K_2 = 0 \Rightarrow S_0 = K_2$

NB. –We expect: $K_1 > S_0$: we expect $\text{Profit}_0 = K_1 - S_0 > 0$.

-But, $K_1 < K_2 = S_0$ if the call is at-the-money.

-We don’t mind it’s a zero-cost collar.

c) $K_1 \leq F \leq K_2$

Proof of $K_1 \leq F$:

$0 = P_{k_1} - C_{k_2} = S_0 - PV(K_1) = PV(F) - PV(K_1)$

$0 \leq PV(F) - PV(K_1)$

$0 \leq F - K_1 \Rightarrow K_1 \leq F$

6) Bear spread:

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Premium = $-P_{k_1} + P_{k_2}$, $k_1 &lt; k_2 $</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$C_{k_1} + C_{k_2}$</td>
</tr>
<tr>
<td>$K_2$</td>
<td></td>
</tr>
<tr>
<td>$S_T$</td>
<td></td>
</tr>
</tbody>
</table>