3. Matrix Algebra

Unit matrices

The unit matrix $I$ of order $n$ is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If $A$ is a square matrix of order $n$, then $AI = IA = A$. Also $I = I^{-1}$.
$I$ is sometimes written as $I_n$ if the order needs to be stated explicitly.

Products

If $A$ is a $(n \times l)$ matrix and $B$ is a $(l \times m)$ then the product $AB$ is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik}B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If $A$ is a matrix, then transpose matrix $A^T$ is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If $A$ is a square matrix with non-zero determinant, then its inverse $A^{-1}$ is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of $A_{ij}$ is $(-1)^{i+j}$ times the determinant of the matrix $A$ with the $j$-th row and $i$-th column deleted.

Determinants

If $A$ is a square matrix then the determinant of $A$, $|A| = A_{11}A_{22}A_{33} \ldots$ is defined by

$$|A| = \sum_{i,j,k} e_{ijk}A_{1i}A_{2j}A_{3k} \ldots$$

where the number of the subscripts is equal to the order of the matrix.

$2 \times 2$ matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1}$$ \hspace{1cm} \text{(if individual inverses exist)}$$

$|AB \ldots N| = |A||B| \ldots |N| \hspace{1cm} \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix $Q$ is a square matrix whose columns $q_i$ form a set of orthonormal vectors. For any orthogonal matrix $Q$,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$
7. Hyperbolic Functions

\[
cosh x = \frac{1}{2} (e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots
\]
valid for all \(x\)

\[
sinh x = \frac{1}{2} (e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots
\]
valid for all \(x\)

\[
cosh ix = \cos x
\]
\[
sinh ix = i \sin x
\]
\[
tanh x = \frac{\sinh x}{\cosh x}
\]
\[
\coth x = \frac{\cosh x}{\sinh x}
\]
\[
cosh^2 x - \sinh^2 x = 1
\]

For large positive \(x\):

\[
cosh x \approx \sinh x \to \frac{e^x}{2}
\]
\[
tanh x \to 1
\]

For large negative \(x\):

\[
cosh x \approx -\sinh x \to \frac{e^{-x}}{2}
\]
\[
tanh x \to -1
\]

Relations of the functions

\[
\sinh x = -\sinh(-x)
\]
\[
\cosh x = \cosh(-x)
\]
\[
\tanh x = -\tanh(-x)
\]
\[
\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}}
\]
\[
\cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}
\]
\[
\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}
\]
\[
\sinh(2x) = 2 \sinh x \cosh x
\]
\[
\cosh(2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x
\]
\[
\sinh(3x) = 3 \sinh x + 4 \sinh^3 x
\]
\[
\cosh(3x) = 4 \cosh^3 x - 3 \cosh x
\]
\[
\tanh(3x) = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}
\]
13. Functions of Several Variables

If \( \phi = f(x, y, z, \ldots) \) then \( \frac{\partial \phi}{\partial x} \) implies differentiation with respect to \( x \) keeping \( y, z, \ldots \) constant.

\[
\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \ldots
\]

and \( \delta \phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \ldots \)

where \( x, y, z, \ldots \) are independent variables. \( \frac{\partial \phi}{\partial x} \) is also written as \( ( \frac{\partial \phi}{\partial x} )_y, \ldots \) or \( \frac{\partial \phi}{\partial x} |_{y, \ldots} \) when the variables kept constant need to be stated explicitly.

If \( \phi \) is a well-behaved function then \( \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \) etc.

If \( \phi = f(x, y) \),

\[
\left( \frac{\partial \phi}{\partial x} \right)_y = \frac{1}{\left( \frac{\partial \phi}{\partial x} \right)_x}, \quad \left( \frac{\partial \phi}{\partial y} \right)_x \left( \frac{\partial \phi}{\partial y} \right)_y \left( \frac{\partial \phi}{\partial x} \right)_x = -1.
\]

Taylor series for two variables

If \( \phi(x, y) \) is well-behaved in the vicinity of \( x = a, y = b \) then it has a Taylor series

\[
\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + \frac{1}{2!} \left( u^2 \frac{\partial^2 \phi}{\partial x^2} + 2uv \frac{\partial^2 \phi}{\partial x \partial y} + v^2 \frac{\partial^2 \phi}{\partial y^2} \right) + \ldots
\]

where \( x = a + u, y = b + v \) and the differential coefficients are evaluated at \( x = a, \ y = b \).

Stationary points

A function \( \phi = f(x, y) \) has a stationary point when \( \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0 \). Unless \( \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0 \), the following conditions determine whether it is a minimum, a maximum or a saddle point.

- **Minimum:** \( \frac{\partial^2 \phi}{\partial x^2} > 0 \), or \( \frac{\partial^2 \phi}{\partial y^2} > 0 \), and \( \frac{\partial^2 \phi}{\partial x \partial y} = 0 \).
- **Maximum:** \( \frac{\partial^2 \phi}{\partial x^2} < 0 \), or \( \frac{\partial^2 \phi}{\partial y^2} < 0 \), and \( \frac{\partial^2 \phi}{\partial x \partial y} = 0 \).
- **Saddle point:** \( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} < \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \)

If \( \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0 \) the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If \( \phi = f(x, y, \ldots) \) and the variables \( x, y, \ldots \) are functions of independent variables \( u, v, \ldots \) then

\[
\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} + \ldots
\]

\[
\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} + \ldots
\]

etc.