Note: By moments we mean the moments about origin or raw moments.

The first four moments about the origin are given by

1. $\mu'_1 = E(X) = \text{Mean}$
2. $\mu'_2 = E(X^2)$
3. $\mu'_3 = E(X^3)$
4. $\mu'_4 = E(X^4)$

Note: $\text{Var}(X) = E(X^2) - [E(X)]^2 = \mu'_2 - \mu'_1^2$ - Second moment - square of the first moment.

Definition 4.2 (Moments about mean or Central moments). The $r^{th}$ moment of a random variable $X$ about the mean $\mu$ is defined as $E[(X - \mu)^r]$ and is denoted by $\mu_r$.

The first four moments about the mean are given by

1. $\mu_1 = E(X - \mu) = E(X) - E(\mu) = \mu - \mu = 0$
2. $\mu_2 = E[(X - \mu)^2] = \text{Var}(X)$
3. $\mu_3 = E[(X - \mu)^3]$
4. $\mu_4 = E[(X - \mu)^4]$

Definition 4.3 (Moments about any point $a$). The $r^{th}$ moment of a random variable $X$ about any point $a$ is defined as $E[(X - a)^r]$ and we denote it by $m'_r$.

The first four moments about a point 'a' are as follows

1. $m'_1 = E(X - a) = E(X) - E(a) = \mu - a$
2. $m'_2 = E[(X - a)^2]$.
3. $m'_3 = E[(X - a)^3]$
4. $m'_4 = E[(X - a)^4]$

Relation between moments about the mean and moments about any arbitrary point $a$

Let $\mu_r$ be the $r^{th}$ moment about mean and $m'_r$ be the $r^{th}$ moment about any point $a$. Let $\mu$ be the mean of $X$. 

Since the total sales $X$ is in thousands of units, the sales between 500 and 1500 units is the event $A$ which stands for $\frac{1}{2} = 0.5 < X < \frac{3}{2} = 1.5$ and the sales over 1000 units is the event $B$ which stands for $X > 1$. $\implies A \cap B = 1 < X < 1.5$

Now

$$P(A) = P(0.5 < X < 1.5) = \int_{0.5}^{1.5} f(x) \, dx$$

$$= \int_{0.5}^{1} x \, dx + \int_{1}^{1.5} (2 - x) \, dx = \frac{3}{4}$$

$$P(B) = P(X > 1) = \int_{1}^{2} f(x) \, dx$$

$$= \int_{1}^{2} (2 - x) \, dx = \frac{1}{2}$$

$$P(A \cap B) = P(1 < X < 1.5) = \int_{1}^{1.5} f(x) \, dx$$

$$= \int_{1}^{1} (2 - x) \, dx = \frac{3}{8}$$

The condition for independent events: $P(A) \cdot P(B) = P(A \cap B)$

Here, $P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$

$\therefore A$ and $B$ are independent events.

Example: 11. If a random variable $X$ has the following probability distribution, find $E(X)$, $E(X^2)$, $Var(X)$, $E(2X + 1)$, $Var(2X + 1)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.3</td>
<td>0.1</td>
<td>0.4</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Hints/Solution: Here $X$ is a discrete RV. $\therefore$

$$E(X) = \sum_{i=-\infty}^{\infty} x_i p(x_i)$$

$$= (-1) \times 0.3 + 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.2$$

$$= -0.3 + 0 + 0.4 + 0.4 = 0.5$$
Now \( E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx \)

\[ = \int_{-\infty}^{0} x^2 \frac{1}{5} e^{-\frac{x}{5}} \, dx \]

\[ = -\frac{1}{5} \left[ (5x^2 + 50x + 250) e^{-\frac{x}{5}} \right]_0^\infty = 50 \]

\[ \therefore \quad \text{Var}(X) = 50 - [5]^2 = 25 \]

**Example: 14.** Find the mean and standard deviation of the distribution

\[ f(x) = \begin{cases} kx(2-x), & \text{when } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \]

**Hints/Solution:** Given that the continuous RV \( X \) whose pdf is given by

\[ f(x) = \begin{cases} kx(2-x), & \text{when } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \]

Since \( f(x) \) is a pdf,

we have \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

\[ i.e. \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{2} f(x) \, dx + \int_{2}^{\infty} f(x) \, dx = 1 \]

\[ i.e. \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{2} kx(2-x) \, dx + \int_{2}^{\infty} 0 \cdot dx = 1 \]

\[ i.e. \quad 0 + k \left[ x^2 - \frac{x^3}{3} \right]_0^2 + 0 = 1 \]

\[ \Rightarrow \quad k = \frac{3}{4} \]
Hints/Solution: We know that
\[
\int_{-\infty}^{\infty} f(x) \, dx = 1
\]
i.e.,
\[
\int_{0}^{\infty} kx^2e^{-x} \, dx = 1
\]
i.e.,
\[
k \int_{0}^{\infty} e^{-x}x^{3-1} \, dx = 1
\]
i.e.,
\[
k \Gamma(3) = 1 \quad \text{[} \because \Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1} \, dx \text{]}
\]
i.e.,
\[
k \cdot 2! = 1 \quad \implies k = \frac{1}{2}
\]
\[
\text{Now, The } r^{\text{th}} \text{ moment is given by}
\]
\[
\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \, dx = \int_{0}^{\infty} x^r kx^2e^{-x} \, dx = k \int_{0}^{\infty} e^{-x}x^{r+3-1} \, dx = \frac{1}{2} \Gamma(r + 3) = \frac{1}{2}(r + 2)!
\]

Now, First Moment \( \mu'_1 = \frac{1}{2}(3)! = 3 \)
Second Moment \( \mu'_2 = \frac{1}{2}(4)! = 12 \)
Third Moment \( \mu'_3 = \frac{1}{2}(5)! = 60 \)
Fourth Moment \( \mu'_4 = \frac{1}{2}(6)! = 360 \)

Example: 20. A random variable \( X \) has the pdf \( f(x) = \frac{1}{2}e^{-\frac{x}{2}}, \ x \geq 0 \). Find the MGF(Moment Generating Function) and hence find its mean and variance.
Hints/Solution: The MGF of $X$ is given by

Given $\mu'_r = (r + 1)!2^r$

$\therefore \mu'_1 = 2!2$
$\mu'_2 = 3!2^2$
$\mu'_3 = 4!2^3$

$\therefore M_X(t) = E(e^{tX}) = 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \cdots + \frac{t^r}{r!}\mu'_r + \cdots$

$= 1 + \frac{t}{1!}2 + \frac{t^2}{2!}3!2^2 + \frac{t^3}{3!}4!2^3 + \cdots$

$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots$

$\therefore M_X(t) = (1 - 2t)^{-2}$.

Now, Differentiating w.r.to $t$, we get

$M'_X(t) = \frac{d}{dt}M_X(t) = -2(1 - 2t)^{-3}(-2) = 6(1 - 2t)^{-4} - 2^2$

Now, First Moment=Mean $E(X) = \mu'_1 = M'_1(0) = 4$

Second Moment $E(X^2) = \mu'_2 = M'_2(0) = 24$

var($X$) $= E(X^2) - [E(X)]^2 = 24 - 16 = 8.$
Find the (i) the value of $A$, (ii) the Distribution Function (CDF) (iii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$, (iv) the probability that in a day the sales is (a) more than 500 kgs (b) less than 500 kgs (c) between 250 and 750 kgs. Ans: $A = \frac{1}{25}$

9. The cumulative distribution function (CDF) of a random variable $X$ is given by

$$F(x) = \begin{cases} 
1 - \frac{4}{x^2}, & \text{when } x > 2 \\
0, & \text{otherwise}
\end{cases}$$

Find the (i) the pdf of $X$, (ii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$ (iii) $P(X < 3)$, $P(3 < X < 5)$.

10. A coin is tossed until a head appears. What is the expected value of the number of tosses? Also find its variance.

11. The pdf of a random variable $X$ is given by

$$f(x) = \begin{cases} 
a + bx, & \text{when } 0 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}$$

Find the (i) the value of $a, b$ if the mean is 1/2, (ii) the variance of $X$ (iii) $P(X > 0.5/X < 0.5)$

12. The first three moments about the origin are 5, 26, 78. Find the first three moments about the value $x=3$. Ans: 2, 5, -48

13. The first two moments about $x=3$ are 1 and 8. Find the mean and variance. Ans: 4, 7

14. The pdf of a random variable $X$ is given by

$$f(x) = \begin{cases} 
k(1 - x), & \text{when } 0 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}$$

Find the (i) the value of $k$, (ii) the $r^{th}$ moment about origin (iii) mean and variance. Ans: $k = 2$

15. An unbiased coin is tossed three times. If $X$ denotes the number of heads appear, find the MGF of $X$ and hence find the mean and variance.

16. Find the MGF of the distribution whose pdf is $f(x) = ke^{-x}$, $x > 0$ and hence find its mean and variance.

17. The pdf of a random variable $X$ is given by

$$f(x) = \begin{cases} 
x, & \text{when } 0 \leq x \leq 1 \\
2 - x, & \text{when } 1 < x \leq 2 \\
0, & \text{otherwise}
\end{cases}$$

For this find the MGF and prove that mean and variance cannot be find using this MGF and then find its mean and variance using expectation.