Question 3. Verify that for all \( n \geq 1 \), the sum of the squares of the first \( 2n \) positive integers is given by the formula

\[
1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n + 1)(4n + 1)}{3}
\]

Solution.

For any integer \( n \geq 1 \), let \( P_n \) be the statement that

\[
1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n + 1)(4n + 1)}{3}
\]

Base Case. The statement \( P_1 \) says that

\[
1^2 = \frac{(1)(2(1) + 1)(4(1) + 1)}{3} = \frac{3(5)}{3} = 5
\]

which is true.

Inductive Step. Fix \( k \geq 1 \), and suppose that \( P_k \) holds, that is,

\[
1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k + 1)(4k + 1)}{3}.
\]

It remains to show that \( P_{k+1} \) holds, that is,

\[
1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2(k + 1))^2 = \frac{(k + 1)(2(k + 1) + 1)(4(k + 1) + 1)}{3}.
\]

\[
1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2(k + 1))^2 = 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k + 2)^2
\]

\[
= \frac{k(2k + 1)(4k + 1)}{3} + (2k + 1)^2 + (2k + 2)^2
\]

\[
= \frac{k(2k + 1)(4k + 1)}{3} + \frac{3(2k + 1)^2 + 3(2k + 2)^2}{3}
\]

\[
= \frac{k(2k + 1)(4k + 1) + 3(2k + 1)^2 + 3(2k + 2)^2}{3}
\]

\[
= \frac{k(8k^2 + 6k + 1) + 3(4k^2 + 4k + 1) + 3(4k^2 + 8k + 4)}{3}
\]

\[
= \frac{(8k^3 + 6k^2 + k) + (12k^2 + 12k + 3) + (12k^2 + 24k + 12)}{3}
\]

\[
= \frac{8k^3 + 30k^2 + 37k + 15}{3}
\]

On the other side of \( P_{k+1} \),

\[
\frac{(k + 1)(2(k + 1) + 1)(4(k + 1) + 1)}{3} = \frac{(k + 1)(2k + 2 + 1)(4k + 4 + 1)}{3}
\]