\[ x \frac{dy}{dx} + y = 2. \]

NB: \( \frac{d}{dx}(xy) = x \frac{dy}{dx} + y \). Integrating both sides we get \( \int \left[ \frac{d}{dx}(xy) \right] dx = \int (x \frac{dy}{dx} + y) dx \)

\[ xy = \int (x \frac{dy}{dx} + y) dx \]

Integrating both sides

\[ \int [x \frac{dy}{dx} + y] dx = \int 2 dx \]

\[ xy = 2x + c \]

\[ y = \frac{2x + c}{x}. \]

Notice that this differential equation is not separable because it’s impossible to factor the expression for \( y' \) as a function of \( x \) times a function of \( y \). But we can still solve the equation by noticing, by the Product Rule, that

\[ xy' + y = (xy)' \]

and so we can rewrite the equation as

\( (xy)' = 2x \)

If we now integrate both sides of this equation, we get

\[ xy = x^2 + c \]

If we had been given the differential equation in the form \( y' + \left(\frac{1}{x}\right)y = \frac{2}{x} \), we would have had to take the preliminary step of multiplying each side of the equation by \( x \).

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides by a suitable function \( \lambda \) called an integrating factor. We try to find \( \lambda \) so that the left side of the ODE, when multiplied by \( \lambda \), becomes the derivative of the product \( \lambda y \):

To solve the linear differential equation \( y' + P(x)y = Q(x) \), multiply both sides by the integrating factor \( I(x) = e^{\int P(x) dx} \) and integrate both sides.

Example
Task
Solve the DEs

1. \[ \frac{1}{x} \frac{dy}{dx} + y = 1. \]
2. \[ \frac{dy}{dx} - 2(x + 1)^3 = \frac{4}{x + 1}y \text{ at } y(0) = 0. \]

Answer

Re-arranging

\[ \frac{dy}{dx} - \frac{4}{x + 1}y = 2(x + 1)^3 \]

\[ \lambda = e^{\int p(x)dx} = e^{\int -4 \frac{1}{x+1}dx} = e^{-4 \ln |x+1|} = e^{\ln(x+1)^{-4}} = \frac{1}{(x+1)^4}. \]

Multiplying by \( \lambda \) we get

\[ \frac{1}{(x+1)^4} \frac{dy}{dx} - \frac{1}{(x+1)^4} \frac{4}{x + 1}y = 2 \frac{1}{x + 1}(x + 1)^3 \]

\[ \frac{d}{dx} \left( \frac{y}{(x+1)^4} \right) = \frac{2}{x + 1} \]

Integrating both sides (LHS is \( \lambda y \)) ALWAYS

\[ \frac{y}{(x+1)^4} = \int \frac{2}{x + 1} \, dx \]

\[ \frac{y}{(x+1)^4} = 2 \ln|x + 1| + c \]
We have \( f(t) = 200e^{0.02t} \). Then

\[
f(300) = 200e^{0.02(300)} \approx 80,686.
\]

There are 80,686 bacteria in the population after 5 hours.

To find when the population reaches 100,000 bacteria, we solve the equation

\[
100,000 = 200e^{0.02t} \\
500 = e^{0.02t} \\
\ln 500 = 0.02t \\
t = \frac{\ln 500}{0.02} \approx 310.73.
\]

The population reaches 100,000 bacteria after 310.73 minutes.

2.

▶ What differential equation does the function \( P(t) \) satisfy? \( \frac{dP(t)}{dt} = kP(t) \)

▶ What is the value of \( k \)? \( k = \text{rate of growth} = 0.009 \)

▶ What is \( P(0) \)? \( P(0) = \text{initial pop. size} = 500 \)

▶ Give a formula for \( P(t) \): \( P(t) = P(0)e^{kt} \)

▶ What will the population be at the end of the year 2050? At the end of the year 2050 we will have \( t = 50 \) and the population will be

\[
P(50) = 500e^{0.009(50)} = 500e^5 \approx 74,206
\]

3.

We have

\[
1,000,000 = Pe^{0.05(40)}
\]

\[
P = 135,335.28.
\]

She must invest $135,335.28 at 5% interest.

4.
2. If some roots are repeated, e.g. \( \lambda_1 \) has multiplicity \( k_1 \), \( \lambda_2 \) has multiplicity \( k_2 \) etc. you must multiply the corresponding exponential with a polynomial of the order \( k_1 - 1, k_2 - 1 \) etc. with arbitrary coefficients. The solution then looks like this:

\[
y = (C_1 + C_2 x + ... C_{k_1} x^{k_1 - 1}) e^{\lambda_1 x} + (D_1 + D_2 x + ... D_{k_2} x^{k_2 - 1}) e^{\lambda_2 x} + ...
\]

Note that in either case we will have the right number (i.e. \( n \)) of the arbitrary constants.

At this level, we restrict our attention to second-order linear homogeneous differential equations with \textbf{constant coefficients} only.

\section*{Second-order linear equations}

A second-order linear differential equation has the form

\[
P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = G(x)
\]

where \( P, Q, R, \) and \( G \) are continuous functions.

In this section we study the case where \( G(x) = 0 \), for all \( x \), in the ODE. Such equations are called \textbf{homogeneous linear} equations. Or we can rewrite:

\[
a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0
\]

(basic form)

where \( a, b, \) and \( c \) are constants.

Replacing \( \frac{d^2 y}{dx} \) with \( m^2 \), \( \frac{dy}{dx} \) with \( m \), and \( y \) with \( 1 \) will result

\[
am^2 + bm + c = 0 \quad \Rightarrow \text{is called "auxiliary quadratic equation"}
\]

or "auxiliary equation".

Thus, the general solution of the 2\textsuperscript{nd}– order linear differential equation
Solve the equation $4y''' + 12y' + 9y = 0$.

**SOLUTION** The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$ (2r + 3)^2 = 0 $$

so the only root is $r = -\frac{3}{2}$. By (10), the general solution is

$$ y = c_1 e^{-3x/2} + c_2 xe^{-3x/2} $$

**Example**

Solve $y''' - 9y'' + 20y = 0$.

**Solution**

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, which can be factored into

$$ (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 $$

The roots are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$; hence the solution is

$$ y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} $$

**Case III: $b^2 - 4ac < 0$**

In this case the roots and of the auxiliary equation are complex numbers.

If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$ y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) $$

**Example**