1.3 The complex plane

Exercise 1. Prove (3.4) and give necessary and sufficient conditions for equality.

Solution. Let $z$ and $w$ be complex numbers. Then

\[ ||z| - |w|| = ||z - w + |w|| \]
\[ \leq ||z - w| + |w| - |w|| \]
\[ = ||z - w|| \]
\[ = ||z - w| \]

Notice that $|z|$ and $|w|$ is the distance from $z$ and $w$, respectively, to the origin while $|z - w|$ is the distance between $z$ and $w$. Considering the construction of the implied triangle, in order to guarantee equality, it is necessary and sufficient that

\[ ||z| - |w|| = ||z - w|| \]
\[ \iff (|z| - |w|)^2 = |z - w|^2 \]
\[ \iff (|z| - |w|)^2 = |z|^2 - 2Re(z\bar{w}) + |w|^2 \]
\[ \iff |z|^2 - 2|z||w| + |w|^2 = |z|^2 - 2Re(z\bar{w}) + |w|^2 \]
\[ \iff |z||w| = Re(z\bar{w}) \]
\[ \iff |z\bar{w}| = Re(z\bar{w}) \]

Equivalently, this is $z\bar{w} \geq 0$. Multiplying both sides by $\frac{w}{|w|}$, we get $z\bar{w} \cdot \frac{w}{|w|} = |w|^2 \cdot \frac{z}{w} \geq 0$ if $w \neq 0$. If $t = \frac{z}{w} = \left(\frac{1}{|w|}\right) \cdot |w|^2 \cdot \frac{z}{w}$. Then $t = 0$ if and only if $z = \bar{w}$.

Exercise 2. Show that equality occurs in (3.3) if and only if $z_k/z_l \geq 0$ for any integers $k$ and $l$, $1 \leq k, l \leq n$, for which $z_l \neq 0$.

Solution. Not available.

Exercise 3. Let $a \in \mathbb{C}$ and $c > 0$ be fixed. Describe the set of points $z$ satisfying

\[ |z - a| - |z + a| = 2c \]

for every possible choice of $a$ and $c$. Now let $a$ be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.

Solution. Not available.

1.4 Polar representation and roots of complex numbers

Exercise 1. Find the sixth roots of unity.

Solution. Start with $z^6 = 1$ and $z = r\mathrm{cis}(\theta)$, therefore $r^6\mathrm{cis}(6\theta) = 1$. Hence $r = 1$ and $\theta = \frac{2k\pi}{6}$ with $k \in \{-3, -2, -1, 0, 1, 2\}$. The following table gives a list of principle values of arguments and the corresponding value of the root of the equation $z^6 = 1$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
<th>$z_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\pi$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{3\pi}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\pi$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{3\pi}{2}$</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\pi$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{3\pi}{2}$</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
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<tr>
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<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{4}$</td>
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<tr>
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<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

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It remains to verify that \( f \circ g \) is uniformly continuous, since we have already shown that \( f \circ g \) is bounded. We have

\[
\rho(f(x)g(x), f(y)g(y)) = |f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \leq M_1 \epsilon_2 + M_2 \epsilon_1,
\]

wherever \( d(x, y) < \min(\delta_1, \delta_2) \). So choosing \( \epsilon = M_1 \epsilon_2 + M_2 \epsilon_1 \) and \( \delta = \min(\delta_1, \delta_2) \), we have \( \forall \epsilon > 0 \exists \delta > 0 \) such that

\[
|f(x)g(x) - f(y)g(y)| < \epsilon
\]

whenever \( d(x, y) < \delta \). Thus, \( f \circ g \) is uniformly continuous and bounded.

**Exercise 4.** Is the composition of two uniformly continuous (Lipschitz) functions again uniformly continuous (Lipschitz)?

**Solution.** Not available.

**Exercise 5.** Suppose \( f : X \to \Omega \) is uniformly continuous; show that if \( \{x_n\} \) is a Cauchy sequence in \( X \) then \( \{f(x_n)\} \) is a Cauchy sequence in \( \Omega \). Is this still true if we only assume that \( f \) is continuous? (Prove or give a counterexample.)

**Solution.** Assume \( f : X \to \Omega \) is uniformly continuous, then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \rho(f(x), f(y)) < \epsilon \) if \( d(x, y) < \delta \). If \( \{x_n\} \) is a Cauchy sequence in \( X \), we have, for every \( \epsilon_1 > 0 \) there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon_1 \) for all \( n, m > N \). By then, by the uniform continuity, we have that

\[
\rho(f(x_n), f(x_m)) < \epsilon \quad \forall n, m \geq N
\]

whenever \( d(x_n, x_m) < \delta \) which tells us that \( \{f(x_n)\} \) is a Cauchy sequence in \( \Omega \). If \( f \) is continuous, the statement is not true. Here is a counterexample: Let \( f(x) = \frac{1}{x} \) which is continuous on \((0, 1)\). The sequence \( x_n = \frac{1}{n} \) is apparently convergent and therefore a Cauchy sequence in \( X \). But \( \{f(x_n)\} = \{f(\frac{1}{n})\} = \{n\} \) is obviously not Cauchy. Note that \( f(x) = \frac{1}{x} \) is not uniformly continuous on \((0, 1)\). To see that pick \( \varepsilon = 1 \). Then there is no \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \). Assume there exists such a \( \delta \). WLOG assume \( \delta < 1 \) since the interval \((0, 1)\) is considered. Let \( y = x + \delta/2 \) and set \( x = \delta/2 \), then

\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y - x}{xy} \right| = \frac{\delta/2}{\delta/2 - \delta} = \frac{1}{\delta} > 1,
\]

that is no matter what \( \delta < 1 \) we choose, we always obtain \( |f(x) - f(y)| > 1 \). Therefore \( f(x) = \frac{1}{x} \) cannot be uniformly continuous.

**Exercise 6.** Recall the definition of a dense set (1.14). Suppose that \( \Omega \) is a complete metric space and that \( f : (D, d) \to (\Omega; \rho) \) is uniformly continuous, where \( D \) is dense in \((X, d)\). Use Exercise 5 to show that there is a uniformly continuous function \( g : X \to \Omega \) with \( g(x) = f(x) \) for every \( x \) in \( D \).

**Solution.** Not available.

**Exercise 7.** Let \( G \) be an open subset of \( \mathbb{C} \) and let \( P \) be a polygon in \( G \) from \( a \) to \( b \). Use Theorems 5.15 and 5.17 to show that there is a polygon \( Q \subset G \) from \( a \) to \( b \) which is composed of line segments which are parallel to either the real or imaginary axes.
Solution. Not available.

Exercise 8. Use Lebesgue’s Covering Lemma (4.8) to give another proof of Theorem 5.15.

Solution. Suppose \( f : X \to \Omega \) is continuous and \( X \) is compact. To show \( f \) is uniformly continuous. Let \( \epsilon > 0 \). Since \( f \) is continuous, we have for all \( x \in X \) there is a \( \delta_x > 0 \) such that \( \rho(f(x), f(y)) < \epsilon/2 \) whenever \( d(x, y) < \delta_x \). In addition, \( X = \bigcup_{x \in X} B(x; \delta_x) \) is an open cover of \( X \). Since \( X \) is by assumption compact (it is also sequentially compact as stated in Theorem 4.9 p. 22), we can use Lebesgue’s Covering Lemma 4.8 p. 21 to obtain a \( \delta > 0 \) such that \( x \in X \) implies that \( B(x, \delta) \subset B(z; \delta_z) \) for some \( z \in X \). More precisely, \( x, y \in B(z; \delta_z) \) and therefore
\[
\rho(f(x), f(z)) \leq \rho(f(x), f(y)) + \rho(f(y), f(z)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
and hence \( f \) is uniformly continuous on \( X \).

Exercise 9. Prove the following converse to Exercise 2.5. Suppose \( (X, d) \) is a compact metric space having the property that for every \( \epsilon > 0 \) and for any points \( a, b \) in \( X \), there are points \( z_0, z_1, \ldots, z_n \) in \( X \) with \( z_0 = a \), \( z_n = b \), and \( d(z_{k-1}, z_k) < \epsilon \) for \( 1 \leq k \leq n \). Then \( (X, d) \) is connected. (Hint: Use Exercise 5.17.)

Solution. Not available.

Exercise 10. Let \( f \) and \( g \) be continuous functions from \( (X, d) \) to \( (\Omega, \rho) \) and let \( D \) be a dense subset of \( X \). Prove that if \( f(x) = g(x) \) for \( x \in D \), then \( f = g \). Use this to show that the function \( g \) obtained in Exercise 6 is unique.

Solution. Not available.

2.6 Uniform convergence

Exercise 1. Let \( \{f_n\} \) be a sequence of uniformly continuous functions from \( (X, d) \) into \( (\Omega, \rho) \) and suppose that \( f = u - \lim f_n \) exists. Prove that \( f \) is uniformly continuous. If each \( f_n \) is a Lipschitz function with constant \( M_n \) and \( \sup M_n < \infty \), show that \( f \) is a Lipschitz function. If \( \sup M_n = \infty \), show that \( f \) may fail to be Lipschitz.

Solution. Not available.
Chapter 3

Elementary Properties and Examples of Analytic Functions

3.1 Power series

Exercise 1. Prove Proposition 1.5.

Solution. Not available.

Exercise 2. Give the details of the proof of Proposition 1.6.

Solution. Not available.

Exercise 3. Prove that \( \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n \) and \( \liminf (a_n + b_n) \geq \liminf a_n + \liminf b_n \) for bounded sequences of real numbers \( \{a_n\} \) and \( \{b_n\} \).

Solution. Let \( r > \limsup_{n \to \infty} a_n \) (we know there are only finitely many by definition) and let \( s > \limsup_{n \to \infty} b_n \) (same here, there are only finitely many by definition). Then \( r + s > a_n + b_n \) for all but finitely many \( n \)’s. This however, implies that

\[ r + s \geq \limsup_{n \to \infty} (a_n + b_n). \]

Since this holds for any \( r > \limsup_{n \to \infty} a_n \) and \( s > \limsup_{n \to \infty} b_n \), we have

\[ \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n. \]

Let \( r < \liminf_{n \to \infty} a_n \) (we know there are only finitely many by definition) and let \( s < \liminf_{n \to \infty} b_n \) (same here, there are only finitely many by definition). Then \( r + s < a_n + b_n \) for all but finitely many \( n \)’s. This however, implies that

\[ r + s \leq \liminf_{n \to \infty} (a_n + b_n). \]

Since this holds for any \( r < \liminf_{n \to \infty} a_n \) and \( s < \liminf_{n \to \infty} b_n \), we have

\[ \liminf_{n \to \infty} (a_n + b_n) \geq \liminf_{n \to \infty} a_n + \liminf_{n \to \infty} b_n. \]

Exercise 4. Show that \( \liminf a_n \leq \limsup a_n \) for any sequence in \( \mathbb{R} \).
Solution. Let \( m = \liminf_{n \to \infty} a_n \) and \( b_n = \inf\{a_n, a_{n+1}, \ldots\} \). Let \( M = \limsup_{n \to \infty} a_n \). Take any \( s > M \). Then, by definition of the \( \limsup \) and \( \liminf \), we obtain that \( a_n < s \) for infinitely many \( n \)'s which implies that \( \limsup_{n \to \infty} b_n = m < s \). This holds for all \( s > M \). But the infimum of all these \( s \)'s is \( M \). Therefore \( m \leq M \) which is

\[
\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n.
\]

Exercise 5. If \( \{a_n\} \) is a convergent sequence in \( \mathbb{R} \) and \( a = \lim a_n \), show that \( a = \liminf a_n = \limsup a_n \).

Solution. Suppose that \( \{a_n\} \) is a convergent sequence in \( \mathbb{R} \) with limit \( a = \lim_{n \to \infty} a_n \). Then by definition, we have: \( \forall \epsilon > 0 \exists N > 0 \) such that \( \forall n \geq N \), we have \( |a_n - a| \leq \epsilon \), that is \( a - \epsilon \leq a_n \leq a + \epsilon \). This means that all but finitely many \( a_n \)'s are \( \leq a + \epsilon \) and \( \geq a - \epsilon \). This shows that

\[
a - \epsilon \leq \liminf_{n \to \infty} a_n \leq a + \epsilon
\]

and

\[
a - \epsilon \leq \limsup_{n \to \infty} a_n \leq a + \epsilon.
\]

By the previous Exercise 4, we also have

\[
a - \epsilon \leq m \leq a + \epsilon.
\]

Hence,

\[
0 \leq M - m \leq 2\epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we obtain \( m = M \). And hence

\[
\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n \leq a + \epsilon,
\]

we obtain

\[
\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = a.
\]

Exercise 6. Find the radius of convergence for each of the following power series: (a) \( \sum_{n=0}^{\infty} a^n z^n \), \( a \in \mathbb{C} \); (b) \( \sum_{n=0}^{\infty} a^n z^n \), \( a \in \mathbb{C} \); (c) \( \sum_{n=0}^{\infty} k^n z^n \), \( k \) an integer \( \neq 0 \); (d) \( \sum_{n=0}^{\infty} n! z^n \).

Solution. a) We have \( \sum_{n=0}^{\infty} a^n z^n = \sum_{k=0}^{\infty} b_k z^n \) with \( b_k = a^k \), \( a \in \mathbb{C} \). We also have,

\[
\limsup_{k \to \infty} |b_k|^{1/k} = \limsup_{k \to \infty} |a|^{|k|/k} = \limsup_{k \to \infty} |a| = |a|.
\]

Therefore, \( R = 1/|a| \), so

\[
R = \begin{cases} 
\frac{1}{|a|}, & a \neq 0 \\
\infty, & a = 0 
\end{cases}
\]

b) In this case, \( b_n = a^n \) where \( a \in \mathbb{C} \).

\[
R = \lim_{n \to \infty} \frac{b_n}{b_{n+1}} = \lim_{n \to \infty} \left| \frac{a^n}{a^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{a^{n+2}}{a^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{1}{a^{n+1}} \right|
\]

\[
= \lim_{n \to \infty} \frac{1}{|a|^{n+1}} = \begin{cases} 
0, & |a| > 1 \\
1, & |a| = 1 \\
\infty, & |a| < 1 
\end{cases}
\]

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(c) Now, \( b_n = k^n \), \( k \) is an integer \( \neq 0 \). We have
\[
R = \limsup_{n \to \infty} |b_n|^{1/n} = \limsup_{n \to \infty} |k^n|^{1/n} = \limsup_{n \to \infty} |k| = |k|.
\]
So
\[
R = \begin{cases} 
\frac{1}{|k|}, & k > 0, k \text{ integer} \\
-\frac{1}{|k|}, & k < 0, k \text{ integer} 
\end{cases}
\]

(d) We can write
\[
\sum_{n=0}^{\infty} z^n = \sum_{k=0}^{\infty} a_k z^k
\]
where
\[
a_k = \begin{cases} 
0, & k = 0 \\
2, & k = 1 \\
1, & k = n!, n \in \mathbb{N}, n > 1 \\
0, & \text{otherwise}
\end{cases}
\]

Thus,
\[
\limsup_{k \to \infty} |a_k|^{1/k} = \limsup_{k \to \infty} |1|^{1/k} = 1.
\]
Therefore \( 1/R = 1 \) which implies \( R = 1 \).

**Exercise 7.** Show that the radius of convergence of the power series
\[
\sum_{n=0}^{\infty} (-1)^n n! z^{n^2 + 1}
\]
is 1, and discuss convergence for \( z = 1, -1, \) and \( i \). (Hint: The \( n \)-th coefficient of this series is not \( (-1)^n / n \).)

**Solution.** As the power series is standard form, then
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n} n!}{n} z^n = \sum_{k=0}^{\infty} a_k z^k
\]
with \( a_k = \begin{cases} 
\frac{(-1)^{n}}{n}, & \text{if } \exists n \in \mathbb{N} \text{ s.t. } k = n(n+1) \\
0, & \text{else}
\end{cases} \)

To find the radius of convergence we use the root criterion and therefore need the estimates
\[
1 \leq \frac{\sqrt[n]{n+1}}{\sqrt[n]{n}} \leq \sqrt[n]{n} \quad \text{for } n \in \mathbb{N}.
\]
The first inequality is immediate from the fact that \( n \geq 1 \) and hence \( n^{1/\sqrt[n]{n}} \geq 1 \). For the second inequality note that
\[
n \leq n^{n+1}
\]
\[
\Leftrightarrow n^{1/\sqrt[n]{n}} \leq n^{1/2}
\]
\[
\Leftrightarrow \sqrt[n]{n} \leq \sqrt[n]{n}.
\]

Using this one obtains
\[
\sqrt[n]{\frac{(-1)^n}{n}} = \frac{1}{\sqrt[n]{n}^{n}} \leq 1
\]
and
\[
\sqrt[n]{\frac{(-1)^n}{n}} = \frac{1}{\sqrt[n]{n}^{n}} \geq \frac{1}{\sqrt[n]{n}}.
\]
Vague memories of calculus classes tell me that $\sqrt{n} \to 1$, thus $\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{a_n} = 1$, i.e. $R = 1$.

If $z = 1$ the series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges with the Leibniz Criterion.

If $z = -1$ we note that the exponents $n(n+1)$ are always even integers and therefore the series is the same as in the previous case of $z = 1$.

Now let $z = i$. The expression $i^{n(n+1)}$ will always be real, so if the series converges at $z = i$, it converges to a real number. We also note that formally

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{n}{n+1}}} = \sum_{n=1}^{\infty} c_n \text{ with } \begin{cases} \frac{1}{n} & \text{if } n \mod 4 \in \{0, 1\} \\ \frac{1}{n} & \text{if } n \mod 4 \in \{2, 3\} \end{cases}$$

Define the partial sums $S_k := \sum_{n=0}^{k} c_n$. We claim that the following chain of inequalities holds

$$0 \leq S_{4k+3} < S_{4k} < S_{4k+2} < S_{4k+1} \leq 1.$$

To verify this, note that

$$S_{4k+3} - S_{4k} = c_{4k+1} + c_{4k+2} + c_{4k+3} = -\frac{16k^2 + 8k - 1}{(4k+1)(4k+2)(4k+3)} < 0 \quad (b)$$

and

$$S_{4k+4} - S_{4k} = c_{4k+3} + c_{4k+4} = \frac{1}{4k+3} + \frac{1}{4k+4} \quad (c)$$

Relation $b)$ is obvious and we know the upper bound $c_1 \geq 1$. To verify the non-negativity constraint. We remark that $\{S_{4k+1}\}_{k \geq 1}, 1 \in \{0, 2, 3\}$ is the bounded and non-decreasing subsequences that converge to some point. Now that $k \in 4N$, the difference between $S_{4k+1}$ and $S_{4k+4}, l, m \in 0, 1, 2, 3$ tends to zero, i.e. all subsequences converge to the same limit. Therefore the power series converges also in the case of $z = i$.

### 3.2 Analytic functions

**Exercise 1.** Show that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

**Solution.** The derivative of $f$ at $z$ is given by

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}, \quad h \in \mathbb{C}$$

provided the limit exist. We have

$$\frac{f(z + h) - f(z)}{h} = \frac{|z + h|^2 - |z|^2}{h} = \frac{(z + h)(\bar{z} + \bar{h}) - z\bar{z}}{h} = \frac{z\bar{h} + \bar{z}h + \bar{h}h - z\bar{z}}{h}$$

$$= \frac{\bar{z} + \bar{h} + z + h}{h} = D.$$ 

If the limit of $D$ exists, it may be found by letting the point $h = (x, y)$ approach the origin $(0,0)$ in the complex plane $\mathbb{C}$ in any manner.

1.) Take the path along the real axes, that is $y = 0$. Then $\frac{\bar{h}}{h} = \frac{h}{h} = 1$ and thus

$$D = \bar{z} + \bar{h} + z = \bar{z} + h + z$$
1. Claim: $\cos(z) \cos(w) - \sin(z) \sin(w) = \cos(z + w)$.
Proof:

$$\cos(z) \cos(w) - \sin(z) \sin(w) = \frac{e^{iz} + e^{-iz}}{2} \frac{e^{iw} + e^{-iw}}{2} - \frac{e^{iz} - e^{-iz}}{2i} \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{1}{4} (e^{iz} e^{iw} + e^{-iz} e^{-iw}) + \frac{1}{4} (e^{iz} e^{iw} + e^{-iz} e^{-iw})$$

$$= \frac{1}{2} e^{iw} + \frac{1}{2} e^{-iw}$$

$$= \cos(z + w).$$

2. Claim: $\sin(z) \cos(w) + \cos(z) \sin(w) = \sin(z + w)$.
Proof:

$$\sin(z) \cos(w) + \cos(z) \sin(w) = \frac{e^{iz} - e^{-iz}}{2i} \frac{e^{iw} + e^{-iw}}{2} + \frac{e^{iz} + e^{-iz}}{2} \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{1}{4i} (e^{iz} e^{iw} - e^{-iz} e^{-iw}) + \frac{1}{4i} (e^{iz} e^{iw} - e^{-iz} e^{-iw})$$

$$= \frac{1}{2i} e^{iw} - \frac{1}{2i} e^{-iw}$$

$$= \sin(z + w).$$

Exercise 8. The $\tan z = \frac{\sin z}{\cos z}$, where is this function defined and analytic?

Solution. Since both $\sin z$ and $\cos z$ are analytic in the entire complex plane, it follows from the discussion in the text following Definition 2.3 that $\tan z = \frac{\sin z}{\cos z}$ is analytic wherever $\cos z \neq 0$. Now, $\cos z = 0$ implies that $z$ is real and equal to an odd multiple of $\frac{\pi}{2}$. Thus let

$$G = \left\{ \frac{(2k + 1)\pi}{2} \mid k \in \mathbb{Z} \right\}.$$

Then $\tan z$ is defined and analytic on $\mathbb{C} - G$. If $z \in G$, then $\cos z = 0$ so $\tan z$ is undefined on the non-extended complex plane.

Exercise 9. Suppose that $z_n, z \in G = \mathbb{C} - \{z : z \leq 0\}$ and $z_n = r_n e^{i\theta_n}, z = r e^{i\theta}$ where $-\pi < \theta, \theta_n < \pi$. Prove that if $z_n \to z$ then $\theta_n \to \theta$ and $r_n \to r$.

Solution. Not available.

Exercise 10. Prove the following generalization of Proposition 2.20. Let $G$ and $\Omega$ be open in $\mathbb{C}$ and suppose $f$ and $h$ are functions defined on $G$, $g : \Omega \to \mathbb{C}$ and suppose that $f(G) \subset \Omega$. Suppose that $g$ and $h$ are analytic, $g'(\omega) \neq 0$ for any $\omega$, that $f$ is continuous, $h$ is one-one, and that they satisfy $h(z) = g(f(z))$ for $z$ in $G$. Show that $f$ is analytic. Give a formula for $f'(z)$.

Solution. Not available.

Exercise 11. Suppose that $f : G \to \mathbb{C}$ is a branch of the logarithm and that $n$ is an integer. Prove that $z^n = \exp(nf(z))$ for all $z$ in $G$. 

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For the real and imaginary part of \( w \) the following equations must hold
\[
\begin{align*}
a &= \frac{1}{2} \left( r + \frac{1}{r} \right) \cos \theta \\
b &= \frac{1}{2} \left( r - \frac{1}{r} \right) \sin \theta.
\end{align*}
\] (3.7)

If \( f(z) = w \) has imaginary part \( b = 0 \) then \( \sin \theta = 0 \) and \( |\cos \theta| = 1 \). Therefore points of the form \( a + ib, a \in [-1, 1], b = 0 \) cannot be in the range of \( f \). For all other points the equations (3.7) can be solved for \( r \) and \( \theta \) uniquely (after restricting the argument to \([-\pi, \pi])\).

Given any value of \( r \in (0, 1) \), the graph of \( f(re^{i\theta}) \) as a function of \( \theta \) looks like an ellipse. In fact from formulas (3.7) we see that \((\frac{a}{\frac{1}{2}r+\frac{1}{2}})^2 + (\frac{b}{\frac{1}{2}r-\frac{1}{2}})^2 = 1\).

If we fix the argument \( \theta \) and let \( r \) vary in \((0, 1)\) it follows from from equation (3.7) that the graph of \( f(re^{i\theta}) \) is a hyperbola and it degenerates to rays if \( z \) is purely real or imaginary. In the case \( \theta \in \{ (2k+1)\pi \mid k \in \mathbb{Z} \} \) the graph of \( f \) in dependence on \( r \) is on the imaginary axis and for \( \theta \in \{ 2k\pi \mid k \in \mathbb{Z} \} \) the graph of \( f(re^{i\theta}) \) is either \((-\infty, -1)\) or \((1, \infty)\). If \( \cos \theta \neq 0 \) and \( \sin \theta \neq 0 \) then \((\frac{a}{\cos \theta})^2 - (\frac{b}{\sin \theta})^2 = 1\).

**Exercise 14.** Suppose that one circle is contained inside another and that they are tangent at the point \( a \). Let \( G \) be the region between the two circles and map \( G \) conformally onto the open unit disk. (Hint: first try \((z-a)^{-1}\).)

**Solution.** Using the hint, define the Möbius transformation \( S = (z-a)^{-1} \) which sends the region \( G \) between two lines. Afterward applying a rotation and then a translation, it is possible to send this region to any region bounded by two parallel lines \( v \). Hence, choose \( S(z) = cz + d \) where \( |c| = 1 \) such that
\[
S(T(G)) = \{ x + iy : 0 < x < \frac{1}{2} \}.
\]

Applying the exponential function to the region yields the right half plane
\[
\exp(S(T(G))) = \{ x + iy : x > 0 \}.
\]

Finally, the Möbius transformation
\[
R(z) = \frac{z - 1}{z + 1}
\]
maps the right half plane onto the unit disk (see page 53). Hence the function \( f \) defined by \( R(\exp(S(T(z)))) \) maps \( G \) onto \( D \) and is the desired conformal mapping (\( f \) is a composition of conformal mappings). Doing some simplifications, we obtain
\[
f(z) = \frac{e^{\pi \tau d} - 1}{e^{\pi \tau d} + 1}
\]
where the constants \( c \) and \( d \) will depend on the circle location.

**Exercise 15.** Can you map the open unit disk conformally onto \( \{ z : 0 < |z| < 1 \} \)?

**Solution.** Not available.

**Exercise 16.** Map \( G = \mathbb{C} \setminus \{ z : -1 \leq z \leq 1 \} \) onto the open unit disk by an analytic function \( f \). Can \( f \) be one-one?

**Solution.** Not available.

**Exercise 17.** Let \( G \) be a region and suppose that \( f : G \to \mathbb{C} \) is analytic such that \( f(G) \) is a subset of a circle. Show that \( f \) is constant.
Exercise 24. Let $T$ be a Möbius transformation, $T \neq \text{id}$. Show that a Möbius transformation $S$ commutes with $T$ if $S$ and $T$ have the same fixed points. (Hint: Use Exercises 21 and 22.)

Solution. Let $T$ and $S$ have the same fixed points. To show $TS = ST$, $T \neq \text{id}$.

\* Suppose $T$ and $S$ have two fixed points, say $z_1$ and $z_2$. Let $M$ be a Möbius transformation with $M(z_1) = 0$ and $M(z_2) = \infty$. Then

$$MSM^{-1}(0) = MSM^{-1}Mz_1 = MSz_1 = Mz_1 = 0$$

and

$$MSM^{-1}(\infty) = MSM^{-1}Mz_2 = MSz_2 = Mz_2 = \infty.$$  

Thus $MSM^{-1}$ is a dilation by exercise 22a) since $MSM^{-1}$ has 0 and $\infty$ as its only fixed points. Similar, we obtain $MTM^{-1}(0) = 0$ and $MTM^{-1}(\infty) = \infty$ and therefore is also a dilation. It is easy to check that dilations commute (define $C(z) = az$, $a > 0$ and $D(z) = bz$, $b > 0$, then $CD(z) = abz = baz = DC(z)$), thus

$$(MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1})$$

$\Rightarrow$ $MTS^{-1} = MST^{-1}$

$\Rightarrow$ $TS = ST$.

\* Suppose $T$ and $S$ have one fixed points, say $z$. Let $M$ be a Möbius transformation with $M(z) = \infty$. Then

$$MSM^{-1}(\infty) = MSM^{-1}Mz = MSz = Mz = \infty.$$  

Thus $MSM^{-1}$ is a translation by exercise 22b) since $MSM^{-1}$ has $\infty$ as its only fixed point. Similar, we obtain $MTM^{-1}(\infty) = \infty$ and therefore is also a translation. It is easy to check that translations commute (define $C(z) = z + 1$ and $D(z) = z + b + a = DC(z)$), thus

$$(MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1})$$

$\Rightarrow$ $MTS^{-1} = MST^{-1}$

$\Rightarrow$ $TS = ST$.

Exercise 25. Find all the abelian subgroups of the group of Möbius transformations.

Solution. Not available.

Exercise 26. 26. (a) Let $GL_2(\mathbb{C}) = \{\text{all invertible } 2 \times 2\text{ matrices with entries in } \mathbb{C}\}$ and let $M$ be the group of Möbius transformations. Define $\varphi : GL_2(\mathbb{C}) \to M$ by $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az + b}{cz + d}$. Show that $\varphi$ is a group homomorphism of $GL_2(\mathbb{C})$ onto $M$. Find the kernel of $\varphi$.

(b) Let $SL_2(\mathbb{C})$ be the subgroup of $GL_2(\mathbb{C})$ consisting of all matrices of determinant 1. Show that the image of $SL_2(\mathbb{C})$ under $\varphi$ is all of $M$. What part of the kernel of $\varphi$ is in $SL_2(\mathbb{C})$?

Solution. a) We have to check that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

then $\varphi(AB) = \varphi(A) \circ \varphi(B)$. A simple calculation shows that this is true. To find the kernel of the group homomorphism we have to find all $z$ such that $\frac{az + b}{cz + d} = z$. This is equivalent to $az + b = cz^2 + dz$ and by comparing coefficients we obtain $b = c = 0$ and $a = d$. Therefore, the kernel is given by $N = \ker(\varphi) = \{\lambda I : \lambda \in \mathbb{C}\}$. Note that the kernel is a normal subgroup of $GL_2(\mathbb{C})$.

b) Restricting $\varphi$ to $SL_2(\mathbb{C})$ still yields a surjective map since for any matrix $A \in GL_2(\mathbb{C})$ both $A$ and the modification $M = \frac{1}{\text{det}A}A$ have the same image and the modification matrix $M$ has by construction determinant 1. The kernel of the restriction is simply $N \cap SL_2(\mathbb{C}) = \{\pm I\}$. 

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0 ≤ t ≤ 1. So \( \gamma_1'(t) = i \) and \( \gamma_2'(t) = -1 \). Therefore,

\[
\int f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_{\gamma_1} |z|^2 \, dz + \int_{\gamma_2} |z|^2 \, dz = \int_0^1 (t^2 + 1)(i) \, dt + \int_0^1 ((1 - t)^2 + 1)(-1) \, dt = i \int_0^1 (t^2 + 1) \, dt - \int_0^1 (t^2 - 2t + 2) \, dt = i \left[ \frac{t^3}{3} + t \right]_0 - i \left[ \frac{t^3}{3} - t^2 + 2t \right]_0 = \frac{4}{3}i - \frac{1}{3} - 1 + 2 = \frac{4}{3} - i = \frac{4}{3}(-1 + i).
\]

**Exercise 9.** Define \( \gamma : [0, 2\pi] \to \mathbb{C} \) by \( \gamma(t) = \exp\left(\int_0^t t \, dt\right) \) where \( n \) is some integer (positive, negative, or zero). Show that \( \int_{\gamma_1} z \, dz = 2\pi i \).

**Solution.** Clearly, \( \gamma(t) = e^{int} \) is continuous and smooth on \( [0, 2\pi] \). Thus

\[
\int_{\gamma_1} z \, dz = \int_0^{2\pi} e^{int} \, dt = \frac{1}{in} \left[ e^{int} \right]_0^{2\pi} = \frac{1}{n} \left[ e^{in2\pi} - 1 \right] = \frac{1}{n} \left[ \cos((n + 1)2\pi) - i \sin((n + 1)2\pi) \right] = \frac{1}{n+1} \left[ 1 - i \cdot 0 - 1 \right] = 0.
\]

**Exercise 10.** Define \( \gamma(t) = e^{it} \) for \( 0 ≤ t ≤ 2\pi \) and find \( \int_{\gamma} z^n \, dz \) for every integer \( n \).

**Solution.** Clearly, \( \gamma(t) = e^{int} \) is continuous and smooth on \( [0, 2\pi] \) (It is the unit circle).

**Case 1:** \( n = -1 \)

\[
\int_{\gamma} z^{-1} \, dz = \int_0^{2\pi} e^{-it} \, dt = i \int_0^{2\pi} \, dt = 2\pi i.
\]

**Case 2:** \( n ≠ -1 \)

\[
\int_{\gamma} z^n \, dz = \int_0^{2\pi} e^{int} \, dt = i \int_0^{2\pi} e^{i(n+1)t} \, dt = i \left[ \frac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi} = \frac{1}{n+1} \left[ e^{i(n+1)2\pi} - 1 \right] = \frac{1}{n+1} \left[ \cos((n + 1)2\pi) - i \sin((n + 1)2\pi) \right] = \frac{1}{n+1} \left[ 1 - i \cdot 0 - 1 \right] = 0.
\]

**Exercise 11.** Let \( \gamma \) be the closed polygon \( [1 - i, 1 + i, -1 + i, -1 - i, 1 - i] \). Find \( \int_{\gamma} \frac{1}{z} \, dz \).
$0 \leq t \leq 2\pi$. Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\overline{B}(0; 2 < r < \infty) \subset \mathbb{C}$. Then,

$$f(-2i) = \frac{1}{2\pi i} \int \frac{f(w)}{w + 2i} dw \iff 1 = \frac{1}{2\pi i} \int \frac{f(w)}{w + 2i} dw$$

$$\iff \int \frac{1}{z + 2i} dz = 2\pi i$$

if $2 < r < \infty$. Similarly,

$$f(2i) = \frac{1}{2\pi i} \int \frac{f(w)}{w - 2i} dw \iff \int \frac{1}{z - 2i} dz = 2\pi i.$$ 

It follows that,

$$\int \frac{z^2 + 1}{z(z^2 + 4)} dz = \frac{1}{4} \int \frac{1}{z} dz + \frac{3}{8} \int \frac{1}{z - 2i} dz + \frac{3}{8} \int \frac{1}{z + 2i} dz$$

$$= \frac{1}{4}(2\pi i) + \frac{3}{8}(2\pi i) + \frac{3}{8}(2\pi i)$$

$$= 2\pi i$$

**Exercise 11.** Find the domain of analyticity of $f(z) = \frac{1}{2i} \log \left(1 + \frac{iz}{1 - iz}\right)$; also, show that $\tan f(z) = z$ (i.e., $f$ is a branch of $\arctan z$). Show that $f(z) = \infty \sum_{k=0}^{\infty} \left(\frac{1}{2k+1}\right) \frac{(-1)^k z^{2k+1}}{2k+1}$, for $|z| < 1$

(Hint: see Exercise III. 3.19.)

**Solution.** Not available.

**Exercise 12.** Show that

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}$$

for some constants $E_2, E_4, \ldots$. These numbers are called Euler’s constants. What is the radius of convergence of this series? Use the fact that $1 = \cos z \sec z$ to show that

$$E_{2n} \left( \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} + \cdots + (-1)^{n-1} \binom{2n}{2} E_2 + (-1)^n \right) = 0.$$ 

Evaluate $E_2, E_4, E_6, E_8$. ($E_{10} = 50521$ and $E_{12} = 2702765$).

**Solution.** It is easily seen that

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}, \quad (\text{for some } E_2, E_4, \ldots),$$
Solution. Assume \( \gamma \) is a closed and rectifiable curve in \( G \) and \( \gamma \sim \sigma_1 \). Since \( \sigma_1(t) \equiv a \) and \( \sigma_2(t) \equiv b \), we surely have that \( \sigma_1 \) and \( \sigma_2 \) are closed and rectifiable curves in \( G \). If we can show that \( \sigma_1 \sim \sigma_2 \), then we get \( \gamma \sim \sigma_2 \), since \( \sim \) is an equivalence relation, that is:

\[
\gamma \sim \sigma_1 \quad \text{and} \quad \sigma_1 \sim \sigma_2 \implies \gamma \sim \sigma_2.
\]

So, now we show \( \sigma_1 \sim \sigma_2 \), that is, we have to find a continuous function \( \Gamma : [0, 1] \times [0, 1] \to G \) such that

\[
\Gamma(s, 0) = \sigma_1(s) = a \quad \text{and} \quad \Gamma(s, 1) = \sigma_2(s) = b, \quad \text{for} \quad 0 \leq s \leq 1
\]

and \( \Gamma(0, t) = \Gamma(1, t) \) for \( 0 \leq t \leq 1 \). Let \( \Gamma(s, t) = tb + (1 - t)a \). Clearly \( \Gamma \) is a continuous function. (it is constant with respect to \( s \) and a line with respect to \( t \)). In addition, it satisfies \( \Gamma(s, 0) = a \forall 0 \leq s \leq 1 \) and \( \Gamma(s, 1) = b \forall 0 \leq s \leq 1 \) and \( \Gamma(0, t) = tb + (1 - t)a = \Gamma(1, t) \forall 0 \leq t \leq 1 \). So \( \sigma_1 \sim \sigma_2 \).

Exercise 2. Show that if we remove the requirement “\( \Gamma(0, t) = \Gamma(1, t) \) for all \( t \)” from Definition 6.1, then the curve \( \gamma_0(t) = e^{2\pi it}, 0 \leq t \leq 1 \), is homotopic to the constant curve \( \gamma_1(t) \equiv 1 \) in the region \( G = \mathbb{C} - \{0\} \).

Solution. Not available.

Exercise 3. Let \( C \) = all rectifiable curves in \( G \) joining \( a \) to \( b \) and show that Definition 6.11 gives an equivalence relation on \( C \).

Solution. Not available.

Exercise 4. Let \( G = \mathbb{C} - \{0\} \) and show that every closed curve \( \gamma \) in \( G \) is homotopic to a closed curve whose trace is contained in \( \{z : |z| = 1\} \).

Solution. Not available.

Exercise 5. Evaluate the integral \( \int \frac{dz}{z^2 + 1} \) where \( \theta(0) = 2\cos 2\theta e^{i\theta} \) for \( 0 \leq \theta \leq 2\pi \).

Solution. A sketch shows that the two zeros of \( z^2 + 1 \) (they are \( \pm i \)) are inside of the closed and rectifiable curve \( \gamma \) (The region \( \{z : |z| = 1\} \)) is a clover with four leaves. Using partial fraction decompositions gives

\[
\int_{\gamma} \frac{dz}{z^2 + 1} = \frac{1}{2i} \int_{\gamma} \frac{1}{z - i} dz - \frac{1}{2i} \int_{\gamma} \frac{1}{z + i} dz
\]

\[
= \pi \frac{1}{2i} \int_{\gamma} \frac{1}{z - i} dz = \pi \frac{1}{2i} \int_{\gamma} \frac{1}{z + i} dz
\]

\[
= n(\gamma; i) - n(\gamma; -i)
\]

\[
= \pi \cdot 0 = 0,
\]

since \( n(\gamma; i) = n(\gamma; -i) \) since \( i \) and \( -i \) are contained in the region generated by \( \gamma \). Hence

\[
\int_{\gamma} \frac{dz}{z^2 + 1} = 0.
\]

Exercise 6. Let \( \gamma(\theta) = \theta e^{i\theta} \) for \( 0 \leq \theta \leq 2\pi \) and \( \gamma(\theta) = 4\pi - \theta \) for \( 2\pi \leq \theta \leq 4\pi \). Evaluate \( \int_{\gamma} \frac{dz}{z^2 + \pi^2} \).

Solution. A sketch reveals that the zero \( -\pi i \) is inside the region and the zero \( \pi i \) is outside the region. Using partial fraction decomposition yields

\[
\int_{\gamma} \frac{dz}{z^2 + \pi^2} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - i\pi} dz - \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z + i\pi} dz
\]

\[
= n(\gamma; i\pi) - n(\gamma; -i\pi) = 0 - 1 = -1,
\]
since \(i\pi\) is not contained in the region generated by \(\gamma\), so \(n(\gamma; i\pi) = 0\) and \(-i\pi\) is contained in the region, so \(n(\gamma; -i\pi) = 1\). Hence,

\[
\int_{\gamma} \frac{dz}{z^2 + \pi^2} = -1.
\]

**Exercise 7.** Let \(f(z) = [(z-\frac{1}{2}) \cdot (z-\frac{1}{2}) \cdot (z+\frac{1}{2}) \cdot (z-\frac{3}{2})]^{-1}\) and let \(\gamma\) be the polygon \([0, 2, 2+i, 2i, 0]\). Find \(\int_{\gamma} f\).

**Solution.** Not available.

**Exercise 8.** Let \(G = \mathbb{C} - \{a, b\}\), \(a \neq b\), and let \(\gamma\) be the curve in the figure below.

(a) Show that \(n(\gamma; a) = n(\gamma; b) = 0\).

(b) Convince yourself that \(\gamma\) is not homotopic to zero. (Notice that the word is “convince” and not “prove”. Can you prove it?) Notice that this example shows that it is possible to have a closed curve \(\gamma\) in a region such that \(n(\gamma; z) = 0\) for all \(z\) not in \(G\) without \(\gamma\) being homotopic to zero. That is, the converse to Corollary 6.10 is false.

**Solution.** Let \(\gamma\) be the path depicted on p. 96. We can write it as a sum of 6 paths. Two of them will be closed and have \(a\) and \(b\) in their unbounded component. Therefore, two integrals will be zero and we will have another 2 pair of non-closed paths. The first pair begins at the leftmost crossing pair and goes around \(a\) in opposite direction. The second pair begins at the middle crossing pair and goes around \(b\) in opposite direction. They also meet at the rightmost crossing pair. If we integrate over the path around \(a\) it is equivalent to evaluate \(\int_{\gamma} \frac{dz}{z-a} \) when \(\gamma_2(t) = a + re^{it}\), \(r \geq 0\) and \(2\pi \leq t \leq 2\pi\) for some \(r > 0\). It is easily seen that we have

\[
\int_{\gamma} \frac{dz}{z-a} = \frac{\pi i}{a-n(\gamma; a)}.
\]

and therefore \(n(\gamma; a) = 0\). Similarly we obtain \(n(\gamma; b) = 0\).

**Exercise 9.** Let \(G\) be a region and let \(\gamma_0\) and \(\gamma_1\) be two closed smooth curves in \(G\). Suppose \(\gamma_0 \sim \gamma_1\), and \(\Gamma\) satisfies (6.2). Also suppose that \(\gamma_0(s) = \Gamma(s, t)\) is smooth for each \(t\). If \(w \in \mathbb{C} - G\) define \(h(t) = n(\gamma_t; w)\) and show that \(h : [0, 1] \to \mathbb{Z}\) is continuous.

**Solution.** Not available.

**Exercise 10.** Find all possible values of \(\int_{\gamma} \frac{dz}{z^2}\), where \(\gamma\) is any closed rectifiable curve in \(\mathbb{C}\) not passing through \(\pm i\).

**Solution.** Let \(\gamma\) be a closed rectifiable curve in \(\mathbb{C}\) not passing through \(\pm i\). Using partial fraction decomposition and the definition of the winding number we obtain

\[
\int_{\gamma} \frac{dz}{z^2} = \frac{1}{2i} \left( \int_{\gamma} \frac{dz}{z-1} - \int_{\gamma} \frac{dz}{z+i} \right) = \frac{1}{2i} \left( 2\pi i \ln(\gamma; i) - 2\pi i \ln(\gamma; -i) \right) = \pi \left( n(\gamma; i) - n(\gamma; -i) \right)
\]

**Exercise 11.** Evaluate \(\int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} dz\) where \(\gamma\) is one of the curves depicted below. (Justify your answer.)

**Solution.** Using Corollary 5.9 p. 86 we have \(\forall a \in G - \{\gamma\}\)

\[
\int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} dz = 2\pi i f^{(k)}(a)n(\gamma; a) \frac{1}{k!}
\]
If \( z = a \), then clearly \( f^{(n)}(a) = 0 \) for \( m - (n - k) > 0 \) and \( f^{(n)}(a) \neq 0 \) for \( m - (n - k) = 0 \). If \( k = 0 \) and \( m = n \), then \( f^{(m)}(a) = f^{(n)}(a) = m!g(a) \neq 0 \) since \( g(a) \neq 0 \) by assumption. Hence, \( f^{(n)}(a) = 0 \) for \( n = 1, 2, \ldots, m - 1 \) and \( f^{(n)}(a) \neq 0 \) for \( n = m \).

Exercise 4. Suppose that \( f : G \to \mathbb{C} \) is analytic and one-one; show that \( f'(z) \neq 0 \) for any \( z \) in \( G \).

Solution. Assume \( f'(a) = 0 \) for some \( a \in G \). Then \( f(a) = \alpha \) where \( \alpha \) is a constant. Define

\[
F(z) = f(z) - \alpha.
\]

Then \( F(a) = f(a) - \alpha = \alpha - \alpha = 0 \) and \( F'(a) = f'(a) = 0 \) by assumption. Hence, \( F \) has \( a \) as a root of multiplicity \( m \geq 2 \).

Then, we can use Theorem 7.4 p. 98 to argue that there is an \( \epsilon > 0 \) and \( \delta > 0 \) such that for \( 0 < |\xi - a| < \delta \), the equation \( f(z) = \xi \) has exactly \( m \) simple roots in \( B(a; \epsilon) \), since \( f \) is analytic in \( B(a; \delta) \) and \( \alpha = f(a) \). Also \( f(z) - \alpha \) has a zero of order \( m \geq 2 \) at \( z = a \).

Since \( f(z) = \xi \) has exactly \( m \) (our case \( m \geq 2 \)) simple roots in \( B(a; \epsilon) \), we can find at least two distinct points \( z_1, z_2 \in B(a; \epsilon) \subset G \) such that

\[
f(z_1) = \xi = f(z_2)
\]

contradicting that \( f \) is \( 1 - 1 \). So \( f'(z) \neq 0 \) for any \( z \) in \( G \).

Exercise 5. Let \( X \) and \( \Omega \) be metric spaces and suppose that \( f: X \to \Omega \) is one-one and onto. Show that \( f \) is an open map if \( f \) is a closed map. (A function \( f \) is a closed map if \( f \) takes closed sets onto closed sets.)

Solution. Not available.

Exercise 6. Let \( P: \mathbb{C} \to \mathbb{R} \) be defined by \( P(z) = \Re z \); show that \( P \) is an open map but is not a closed map.

(Hint: Consider the set \( F = \{ z : \Im z = (\Re z)^{-1} \text{ and } \Re z \neq 0 \} \).

Solution. Not available.

Exercise 7. Use Theorem 7.2 to give another proof of the Fundamental Theorem of Algebra.

Solution. Not available.

4.8 Goursat’s Theorem

No exercises are assigned in this section.
Chapter 5

Singularities

5.1 Classification of singularities

Exercise 1. Each of the following functions \( f \) has an isolated singularity at \( z = 0 \). Determine its nature; if it is a removable singularity define \( f(0) \) so that \( f \) is analytic at \( z = 0 \); if it is a pole define the singular part; if it is an essential singularity determine \( f(\{z : 0 < |z| < \delta\}) \) for arbitrarily small values of \( \delta \).

(a) \( f(z) = \frac{\sin z}{z} \);
(b) \( f(z) = \frac{\cos z}{z} \);
(c) \( f(z) = \frac{\cos z - 1}{z} \);
(d) \( f(z) = e^{z-1} \);
(e) \( f(z) = \frac{\log(z + 1)}{z^2} \);
(f) \( f(z) = \frac{\cos(z^{-1})}{z^{-1}} \);
(g) \( f(z) = \frac{z^2 + 1}{z(z - 1)} \);
(h) \( f(z) = \frac{1 - e^z}{z} \);
(i) \( f(z) = z \sin \frac{1}{z} \).
Chapter 6

The Maximum Modulus Theorem

6.1 The Maximum Principle

Exercise 1. Prove the following Minimum Principle. If $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on $G$, then either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial G$. (See Exercise IV 3.6.)

Solution. Since $f \in C(\bar{G})$ we have $|f| \in C(\bar{G})$. Hence there is $a \in \partial G$ such that $|f(a)| \leq |f(z)| \forall z \in \bar{G}$.

If $a \in \partial G$, then $|f|$ assumes its minimum value on $\partial G$ and we are done. Otherwise, if $a \notin \partial G$, then $a \in G$ and we can write $G = \bigcup A_i$ where the components of $G$ which contain $a$ are $A_i$ for some $i$. But each $A_i$ is a region, so we can use Exercise IV 3.6 which yields either $f(a) = 0$ or $f$ is constant. But $f$ was assumed to be non-constant, so $f$ has to have a zero in $G$. Therefore, either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial G$.

Exercise 2. Let $G$ be a bounded region and suppose $f$ is continuous on $G$ and analytic on $G$. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all $z$ on the boundary of $G$ then either $f$ is a constant function or $f$ has a zero in $G$.

Solution. Assume there is a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial G$. According to the Maximum Modulus Theorem (Version 2), we get

$$\max_{z \in \bar{G}} |f(z)| = \max_{z \in \partial G} |f(z)| = c.$$ 

So

$$|f(z)| \leq c, \quad \forall z \in \bar{G}. \tag{6.1}$$

Since $f \in C(G)$ which implies $f \in C(\bar{G})$ and hence there exists an $a \in G$ such that

$$|f(a)| \leq |f(z)| \leq c,$$

then by Exercise IV 3.6 either $f$ is constant or $f$ has a zero in $G$.

Exercise 3. (a) Let $f$ be entire and non-constant. For any positive real number $c$ show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \leq c\}$.

(b) Let $p$ be a polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of $p$. (Hint: Use
Exercise 2.
(c) If \( p \) is a polynomial and \( c > 0 \) show that \( \{z : |p(z)| = c\} \) is the union of a finite number of closed paths.

Discuss the behavior of these paths as \( c \to \infty \).

**Solution.** Not available.

Exercise 4. Let \( 0 < r < R \) and put \( A = \{z : r \leq |z| \leq R\} \). Show that there is a positive number \( \epsilon > 0 \) such that for each polynomial \( p \),

\[
\sup \{|p(z) - z^{-1}| : z \in A\} \geq \epsilon
\]

This says that \( z^{-1} \) is not the uniform limit of polynomials on \( A \).

**Solution.** Not available.

Exercise 5. Let \( f \) be analytic on \( \overline{B(0; R)} \) with \( |f(z)| \leq M \) for \( |z| \leq R \) and \( |f(0)| = a > 0 \). Show that the number of zeros of \( f \) in \( B(0; \frac{1}{3}R) \) is less than or equal to \( \frac{1}{\log 2} \log \left( \frac{M}{a} \right) \). Hint: If \( z_1, \ldots, z_n \) are the zeros of \( f \) in \( B(0; \frac{1}{3}R) \); consider the function

\[
g(z) = f(z) \left[ \prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right)^{-1} \right],
\]

and note that \( g(0) = f(0) \). (Notation: \( \prod_{k=1}^{n} a_k = a_1 a_2 \ldots a_n \).)

**Solution.** Not available.

Exercise 6. Suppose both \( f \) and \( g \) are analytic on \( B(0; R) \) and \( |f(z)| = |g(z)| \) for \( |z| = R \). Show that if neither \( f \) nor \( g \) vanishes in \( B(0; R) \) then there is a constant \( A \geq 0 \) such that \( f = \lambda g \).

**Solution.** Not available.

Exercise 7. Let \( f \) be analytic in the disk \( B(0; R) \) and for \( 0 \leq r < R \) define \( A(r) = \max \{\Re f(z) : |z| = r\} \). Show that unless \( f \) is a constant \( A(r) \) is a strictly increasing function of \( r \).

**Solution.** Not available.

Exercise 8. Suppose \( G \) is a region, \( f : G \to \mathbb{C} \) is analytic, and \( M \) is a constant such that whenever \( z \) is on \( \partial_\infty G \) and \( \{z_n\} \) is a sequence in \( G \) with \( z = \lim z_n \) we have \( \lim \sup |f(z_n)| \leq M \). Show that \( |f(z)| \leq M \) for each \( z \) in \( G \).

**Solution.** We need to show

\[
\lim_{z \to u} \sup |f(z)| \leq M \quad \forall u \in \partial_\infty G.
\]

Then we can use the Maximum Modulus Theorem (Version 3). Instead of showing that

\[
\lim_{z \to u} |f(z_n)| \leq M \Rightarrow \lim_{z \to u} |f(z)| \leq M,
\]

we show the contrapositive, that is

\[
\lim_{z \to u} |f(z)| > M \Rightarrow \lim_{z \to u} |f(z_n)| > M.
\]

So assume \( \lim \sup z \to u |f(z)| > M \). But this implies \( \lim \sup z \to u |f(z_n)| > M \) since \( z_n \to z \) as \( z \to \infty \), that is \( z_n \) gets arbitrarily close to \( z \) and since \( f \) is analytic, that is continuous, we have \( f(z_n) \) gets arbitrarily close to \( f(z) \).
Exercise 6. Prove Hardy’s Theorem: If \( f \) is analytic on \( \overline{B}(0; R) \) and not constant then
\[
I(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta
\]
is strictly increasing and \( \log I(r) \) is a convex function of \( \log r \). Hint: If \( 0 < r_1 < r < r_2 \) find a continuous function \( \varphi : [0, 2\pi] \to \mathbb{C} \) such that \( \varphi(\theta)f(re^{i\theta}) = |f(re^{i\theta})| \) and consider the function \( F(z) = \int_0^{2\pi} f(zre^{i\theta}) \varphi(\theta) \, d\theta \). (Note that \( r \) is fixed, so \( \varphi \) may depend on \( r \).)

Solution. Not available.

Exercise 7. Let \( f \) be analytic in \( \text{ann}(0; R_1, R_2) \) and not identically zero; define
\[
I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta.
\]
Show that \( \log I_2(r) \) is a convex function of \( \log r \), \( R_1 < r < R_2 \).

Solution. Not available.

6.4 The Phragmén-Lindelöf Theorem

Exercise 1. In the statement of the Phragmén-Lindelöf Theorem, the requirement that \( G \) be simply connected is not necessary. Extend Theorem 4.1 to regions \( G \) with the property that for each \( z \) in \( \partial\omega \) there is a sphere \( V \) in \( \mathbb{C}_\infty \) centered at \( z \) such that \( V \cap G \) is simply connected. Give some examples of regions that are not simply connected but have this property or some which don’t.

Solution. Not available.

Exercise 2. In Theorem 4.5 suppose there are bounded analytic functions \( \varphi_1, \varphi_2, \ldots, \varphi_n \) on \( G \) that never vanish and \( \partial\omega \) such that condition (a) is satisfied and condition (b) is also satisfied for each \( \varphi_k \) and \( B_k \). Prove that \( |f(z)| \leq M \) for all \( z \) in \( G \).

Solution. We have \( \varphi_1, \ldots, \varphi_n \in A(G) \), bounded and \( \varphi_k \neq 0 \) by assumption. Hence, \( |\varphi_k(z)| \leq k_k \) for each \( z \) in \( G \). Further a) for every \( a \), \( \limsup_{z \to a} |f(z)| = M \) and b) for every \( b \) in \( B_k \) and \( \eta_k > 0 \), \( \limsup_{z \to b} |f(z)||\varphi_k(z)|^{\eta_k} \leq M \forall k = 1, \ldots, n \). Also because \( G \) is simply connected, there is an analytic branch of \( \log \varphi_k(z) \) on \( G \) for each \( k = 1, \ldots, n \) (Corollary IV 6.17). Hence, \( g_k(z) = \exp(\eta_k \log \varphi_k(z)) \) is an analytic branch of \( \varphi_k(z)^{\eta_k} \) for \( \eta_k > 0 \) and \( |g_k(z)| = |\varphi_k(z)|^{\eta_k} \forall k = 1, \ldots, n \). Define \( F: G \to \mathbb{C} \) by
\[
F(z) = f(z) \prod_{k=1}^{n} g_k(z)k_k^{-\eta_k};
\]
then \( F \) is analytic on \( G \) and
\[
|F(z)| \leq |f(z)| \prod_{k=1}^{n} |\varphi_k(z)|^{k_k}k_k^{-\eta_k} \leq |f(z)| \prod_{k=1}^{n} k_k^{-\eta_k} |\varphi_k(z)| = |f(z)| \forall z \in G.
\]

Hence,
\[
\limsup_{z \to a} |f(z)| = f(a) \begin{cases} M, & a \in A \\ M_k^{-\eta_k}, & a \in B_k \forall k \end{cases}
\]
by definition. Then, clearly $\exists \delta > 0$ such that $\lim_{r \to 0^+} \sup \{|f(z)| : z \in G \cap B(a;r)\} > M + \delta$. Let $\{z_n\}$ be a sequence with $\lim_{n \to \infty} a = a$ and let $r_n = 2|z_n - a|$. Obviously, we have $\lim_{n \to \infty} r_n = 0$. Next, consider the sequence $\alpha_n = \sup\{|f(z)| : z \in F \cap B(a;r_n)\}$. Then $\lim_{n \to \infty} \alpha_n > M + \delta$. In addition, $\{\alpha_n\}$ is a nonincreasing sequence, therefore $\lim_{n \to \infty} \alpha_n > M + \delta \forall n$. Thus, $\exists y_n \in G \cap B(a;r_n)$ such that $|f(y_n)| > M + \delta$. Taking $\limsup$ in this inequality yields

$$\limsup_{n \to \infty} |f(y_n)| \geq M + \delta,$$

that is

$$\limsup_{n \to \infty} |f(y_n)| \geq M$$

where $\lim_{n \to \infty} y_n = a$, which gives “not (6.5)”. Hence (6.5) implies (6.6).

**Exercise 5.** Let $f : G \to \mathbb{C}$ be analytic and suppose that $G$ is bounded. Fix $z_0 \in \partial G$ and suppose that $\limsup_{z \to w} |f(z)| \leq M$ for $w$ in $\partial G$, $w \neq z_0$. Show that if $\lim_{z \to z_0} |z - z_0|^\alpha f(z) = 0$ for every $\epsilon > 0$ then $|f(z)| \leq M$ for every $z$ in $\partial G$. (Hint: If $a \notin G$, consider $\varphi(z) = (z - z_0)(z - a)^{-1}$.)

**Solution.** Not available.

**Exercise 6.** Let $G = \{z : \Re z > 0\}$ and let $f : G \to \mathbb{C}$ be an analytic function with $\limsup_{z \to w} |f(z)| \leq M$ for $w$ in $\partial G$, and also suppose that for every $\epsilon > 0$,

$$\lim_{r \to \infty} \sup \{|f(z)| : |z| = r, 0 < |\theta| < \frac{\pi}{2}\} = 0.$$

*Show that $|f(z)| \leq M$ on $a \notin \partial G$.***

**Solution.** Not available.

**Exercise 7.** Let $G = \{z : \Re z > 0\}$ and let $f : G \to \mathbb{C}$ be analytic such that $f(1) = 0$ and such that $\limsup_{z \to w} |f(z)| \leq 1$ for $w$ in $\partial G$. Also, suppose that for every $\delta$, $0 < \delta < 1$, there is a constant $P$ such that

$$|f(z)| \leq P \exp(|z|^{1-\delta}).$$

*Prove that

$$|f(z)| \leq M \left[\left(1-x\right)^2 + y^2\right]^{1/2},$$

$$(1 + x)^2 + y^2.$$*

**Hint:** Consider $f(z) \left(\frac{x + i}{z}\right)$.

**Solution.** Not available.
We also have
\[ |h(z)| \leq 1 \quad \forall z \in D \]
since \( h : D \to D \). Thus, the hypothesis of Schwarz’s Lemma are satisfied, and hence, we get
\[ |h'(0)| \leq 1. \]

We have
\[
h'(z) = \left[ g(f(g^{-1}(z))) \right]' = g'(f(g^{-1}(z)))f'(g^{-1}(z))\left(g^{-1}(z)' \right)
\]
\[
= g'(f(g^{-1}(z)))f'(g^{-1}(z)) \frac{1}{g'(g^{-1}(z))}
\]
where the last step follows from Proposition 2.20 provided \( g'(g^{-1}(z)) \neq 0 \). So
\[
h'(0) = g'(f(0))f'(0) \frac{1}{g'(0)} = g'(a)f'(a) \frac{1}{g'(a)} = f'(a)
\]
\((g'(a) > 0 \text{ by assumption})\). Therefore
\[ |h'(0)| = |f'(a)| \leq 1. \]
Thus, we have shown that
\[ |f'(a)| \leq 1. \]

**Exercise 6.** Let \( G_1 \) and \( G_2 \) be simply connected, neither of which is the whole plane. Let \( f \) be a one-one analytic mapping of \( G_1 \) onto \( G_2 \) and \( \alpha \in G_2 \) and put \( \alpha = f(\alpha) \). Prove that for any one-one analytic map \( h : G_1 \to G_2 \) with \( h(\alpha) = \alpha \) it follows that \( |h'(\alpha)| \leq |f'(\alpha)| \). Suppose \( h \) is not assumed to be one-one; what can be said?

**Solution.** Define the function \( F(z) = f^{-1}(h(z)) \). This is well-defined since \( f : G_1 \to G_2 \) and \( h : G_1 \to G_2 \). Clearly \( F(z) \) is analytic since \( f \) and \( h \) are analytic, \( F(z) \) is one-one and \( F : G_1 \to G_1 \) by construction. We have
\[ F(\alpha) = f^{-1}(h(\alpha)) = f^{-1}(\alpha) = \alpha. \]
Thus by Exercise 5, we get
\[ |F'(\alpha)| \leq 1. \]
We have
\[
F'(z) = \left( f^{-1}(h(z)) \right)' = f'(h(z))h'(z) = \frac{1}{f'(f^{-1}(h(z)))}h'(z)
\]
where the last step follows by Proposition 2.20 provided \( f'(f^{-1}(h(z))) \neq 0 \). So
\[
F'(\alpha) = \frac{1}{f'(f^{-1}(h(\alpha)))}h'(\alpha) = \frac{h'(\alpha)}{f'(f^{-1}(\alpha))} = \frac{h'(\alpha)}{f'(\alpha)}
\]
which is well-defined since \( f'(\alpha) \neq 0 \) by assumption (\( f \) is one-one). Since
\[ \frac{|h'(\alpha)|}{|f'(\alpha)|} \leq 1 \]
iff A has a limit point in G.

(d) Let \( a \in G \) and \( J = J(\{a\}) \). Show that \( J \) is a maximal ideal.

(e) Show that every maximal ideal in \( H(G) \) is a prime ideal.

(f) Give an example of an ideal which is not a prime ideal.

**Solution.** Not available.

**Exercise 12.** Find an entire function \( f \) such that \( f(n + in) = 0 \) for every integer \( n \) (positive, negative or zero). Give the most elementary example possible (i.e., choose the \( p_n \) to be as small as possible).

**Solution.** Not available.

**Exercise 13.** Find an entire function \( f \) such that \( f(m + in) = 0 \) for all possible integers \( m, n \). Find the most elementary solution possible.

**Solution.** Not available.

### 7.6 Factorization of the sine function

**Exercise 1.** Show that \( \cos \pi z = \prod_{n=1}^{\infty} \left[ 1 - \frac{4z^2}{(2n-1)^2} \right] \).

**Solution.** We know by the double-angle identity of sine \( \sin(2\pi z) = 2 \sin(\pi z) \cos(\pi z) \) (this is proved easily by using the definition) or \( \sin(2\pi z) = 2 \sin(\pi z) \cos(\pi z) \). Also, we know \( \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \), we obtain

\[
\sin(2\pi z) = 2 \sin(\pi z) \cos(\pi z)
\]

\[
\Leftrightarrow 2\pi z \prod_{m=1}^{\infty} \left( 1 - \frac{4z^2}{(2m)^2} \right) \prod_{n=1}^{\infty} \left( 1 - \frac{4z^2}{(2m-1)^2} \right) \cos(\pi z)
\]

where the last statement follows by splitting the product into a product of the even and odd terms (rearrangement of the terms is allowed). Hence

\[
2\pi z \prod_{n=1}^{\infty} \left( 1 - \frac{4z^2}{(2n)^2} \right) \prod_{m=1}^{\infty} \left( 1 - \frac{4z^2}{(2m-1)^2} \right) = 2\pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \cos(\pi z)
\]

\[
\Leftrightarrow 2\pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \prod_{m=1}^{\infty} \left( 1 - \frac{4z^2}{(2m-1)^2} \right) = 2\pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \cos(\pi z).
\]

Thus,

\[
\cos(\pi z) = \prod_{n=1}^{\infty} \left( 1 - \frac{4z^2}{(2n-1)^2} \right).
\]

**Exercise 2.** Find a factorization for \( \sinh z \) and \( \cosh z \).
Thus,

\[
\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_{1}^{\infty} e^{-t}t^{z-1} \, dt \tag{7.12}
\]

Claim 1: \(\Gamma(z)\) given by (7.12) is the analytic continuation of (7.11), that is \(\Gamma(z)\) given by (7.12) is defined for all \(z \in \mathbb{C} - \{0, -1, -2, \ldots\}\).

Proof of Claim 1: We know from the book that \(\Psi(z)\) is analytic for \(\text{Re}(z) > 0\).

Claim 2: \(\Psi(z)\) is analytic for \(\text{Re}(z) \leq 0\). Thus \(\Psi(z)\) is analytic on \(\mathbb{C}\).

Proof of Claim 2: Assume \(\text{Re}(z) \leq 0\). Then

\[
|e^{-1}| = t^{\text{Re}(z)-1} \leq t^{-1}.
\]

But since \(e^{-1}t^{\text{Re}(z)-1} \to 0\) as \(t \to \infty\), there exists a constant \(C > 0\) such that \(t^{\text{Re}(z)-1} \leq Ce^{\frac{1}{t}}\) when \(t \geq 1\). Hence, we have

\[
|e^{-1}| \leq |e^{-1}| = e^{-1}t^{\text{Re}(z)-1} \leq e^{-1}Ce^{\frac{1}{t}} = Ce^{-\frac{1}{t}}
\]

and therefore \(Ce^{-\frac{1}{t}}\) is integrable on \((1, \infty)\). By Fubini’s Theorem for any \(\{\gamma\} \subset G = \{ \text{Re}(z) \leq 0\}\),

\[
\int_{\gamma} \int_{1}^{\infty} e^{-1}t^{z-1} \, dt \, dz = \int_{1}^{\infty} \int_{\gamma} e^{-1}t^{z-1} \, dz \, dt
\]

which implies

\[
\Psi(z) = \int_{1}^{\infty} e^{-1}t^{z-1} \, dt \in H(C).
\]

Thus, Claim 2 is proved.

It remains to show that

\[
\Phi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}
\]

is analytic on \(\mathbb{C} - \{0, -1, -2, \ldots\}\). Note that \(\Phi(z)\) is uniformly and absolutely convergent as a series in any closed domain which contains none of the points 0, -1, -2, \ldots and thus provides the analytic continuation of \(\Phi(z)\). Since

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_{1}^{\infty} e^{-1}t^{z-1} \, dt
\]

is analytic and we know (Theorem 7.15 p. 180)

\[
\Gamma(z) = \int_{0}^{\infty} e^{-1}t^{z-1} \, dt
\]

for \(\text{Re}(z) > 0\), we get

\[
\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_{1}^{\infty} e^{-1}t^{z-1} \, dt
\]

is the analytic continuation of (7.11) for \(z \in \mathbb{C} - \{0, -1, -2, \ldots\}\).
8.3 Mittag-Leffler’s Theorem

Exercise 1. Let \( G \) be a region and let \( \{a_n\} \) and \( \{b_m\} \) be two sequences of distinct points in \( G \) without limit points in \( G \) such that \( a_n \neq b_m \) for all \( n, m \). Let \( S_n(z) \) be a singular part at \( a_n \) and let \( p_m \) be a positive integer. Show that there is a meromorphic function \( f \) on \( G \) whose only poles and zeros are \( \{a_n\} \) and \( \{b_m\} \) respectively, the singular part at \( z = a_n \) is \( S_n(z) \), and \( z = b_m \) is a zero of multiplicity \( p_m \). (Note that the zeros do not cancel the poles since by assumption \( a_n \neq b_m \forall n, m \)).

Solution. Let \( G \) be a region and let \( \{b_m\} \) be a sequence of distinct points in \( G \) with no limit point in \( G \); and let \( \{p_m\} \) be a sequence of integers. By Theorem 5.15 p.170 there is an analytic function \( g \) defined on \( G \) whose only zeros are at the points \( b_m \); furthermore, \( b_m \) is a zero of \( g \) of multiplicity \( p_m \).

Since \( g \in H(g) \) and \( \{a_n\} \in G \), \( g \) has a Taylor series in a neighborhood \( B(a_n; R_n) \) of each \( a_n \), that is

\[
g_n(z) = \sum_{k=0}^{\infty} \alpha_k(z-a_n)^k \in B(a_n; R_n)
\]

where \( \alpha_k = \frac{1}{k!} g^{(k)}(a_n) \). Goal: Try to use this series to create a singular part \( r_n(z) \) at \( a_n \) such that \( r_n(z)g_n(z) = s_n(z) \) or

\[
r_n(z) \sum_{k=0}^{\infty} \alpha_k(z-a_n)^k = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z-a_n)^j} \iff \sum_{k=0}^{m_n} \frac{A_{jn}}{(z-a_n)^j} = \sum_{k=0}^{\infty} \alpha_k r_n(z)(z-a_n)^k.
\]

Claim:

\[
r_n(z) = \sum_{j=1}^{m_n} A_{jn}. \quad (8.1)
\]

Proof of the claim

\[
\sum_{j=1}^{\infty} B_{jk}(z-a_n)^{-j-k} = \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk}(z-a_n)^{-j-k}(z-a_n)^k = \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk}(z-a_n)^{-j} = \sum_{j=1}^{m_n} (z-a_n)^{-j} \sum_{k=0}^{\infty} \alpha_k B_{jk} = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z-a_n)^j}
\]

where the last step follows by choosing

\[
\sum_{k=0}^{\infty} \alpha_k B_{jk} = A_{jn}.
\]

Since \( G \) is a region, \( G \) is open. Let \( \{a_n\} \) be a sequence of distinct points without a limit point in \( G \) and such that \( a_n \neq b_m \) for all \( n, m \). Let \( \{r_n(z)\} \) be the sequence of rational functions given by

\[
r_n(z) = \sum_{j=1}^{m_n} \frac{B_{jk}}{(z-a_n)^{j+k}}
\]

(see (8.1)). By Mittag-Leffler’s Theorem, there is a meromorphic function \( h \) on \( G \) whose poles are exactly the points \( \{a_n\} \) and such that the singular part of \( h \) at \( a_n \) is \( r_n(z) \).

Set \( f = g \cdot h \). Then by construction \( f \) is the meromorphic function on \( G \) whose only poles and zeros are \( \{a_n\} \) and \( \{b_m\} \) respectively, the singular part at \( z = a_n \) is \( S_n(z) \), and \( z = b_m \) is a zero of multiplicity \( p_m \). (Note that the zeros do not cancel the poles since by assumption \( a_n \neq b_m \forall n, m \)).

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Solution. Not available.

Exercise 4. Let \( G \) be a simply connected region and let \( \Gamma \) be its closure in \( \mathbb{C}_\infty \); \( \partial_\infty G = \Gamma - G \). Suppose there is a homeomorphism \( \varphi \) of \( \Gamma \) onto \( D^- (\{ z : |z| < 1 \}) \) such that \( \varphi \) is analytic on \( G \).
(a) Show that \( \varphi(G) = D \) and \( \varphi(\partial_\infty G) = \partial D \).
(b) Show that if \( f : \partial_\infty G \to \mathbb{R} \) is a continuous function then there is a continuous function \( u : \Gamma \to \mathbb{R} \) such that \( u(z) = f(z) \) for \( z \) in \( \partial_\infty G \) and \( u \) is harmonic in \( G \).
(c) Suppose that the function \( f \) in part (b) is not assumed to be continuous at \( \infty \). Show that there is a continuous function \( u : \mathbb{C}_\infty \to \mathbb{R} \) such that \( u(z) = f(z) \) for \( z \) in \( \partial G \) and \( u \) is harmonic in \( G \) (see Exercise 2).

Solution. Not available.

Exercise 5. Let \( G \) be an open set, \( a \in G \), and \( G_0 = G - \{ a \} \). Suppose that \( u \) is a harmonic function on \( G_0 \) such that \( \lim_{z\to a} u(z) \) exists and is equal to \( A \). Show that if \( U : G \to \mathbb{R} \) is defined by \( U(z) = u(z) \) for \( z \neq a \) and \( U(a) = A \) then \( U \) is harmonic on \( G \).

Solution. Not available.

Exercise 6. Let \( f : \{ z : \text{Re } z = 0 \} \to \mathbb{R} \) be a bounded continuous function and define \( u : \{ z : \text{Re } z > 0 \} \to \mathbb{R} \) by
\[
u(x + iy) = \frac{1}{\pi} \int_0^\infty \frac{xf(it)}{x^2 + (y - t)^2} \, dt.
\]
Show that \( u \) is a bounded harmonic function on the right half-plane \( \{ x + iy : x > 0 \} \). What can you say about the behavior of \( v(r) = u(1 + i \frac{r}{2}) \)? What about \( v(r \theta) \) as \( r \to 1^- \) and \( \theta \to 0^- \)?

Solution. Not available.

Exercise 7. Let \( D = \{ z : |z| < 1 \} \). Suppose \( f : \partial D \to \mathbb{R} \) is continuous except for a jump discontinuity at \( z = 1 \). Define \( u : D \to \mathbb{R} \) by (2.5). Show that \( u \) is harmonic. Let \( v \) be a harmonic conjugate of \( u \). What can you say about the behavior of \( v(r) = u(re^i) \) as \( r \to 0^- \)?

Solution. Not available.

10.3 Subharmonic and superharmonic functions

Exercise 1. Which of the following functions are subharmonic? superharmonic? harmonic? neither subharmonic nor superharmonic? (a) \( \varphi(x,y) = x^2 + y^2 \); (b) \( \varphi(x,y) = x^2 - y^2 \); (c) \( \varphi(x,y) = x^2 + y^2 \); (d) \( \varphi(x,y) = x^2 - y^2 \); (e) \( \varphi(x,y) = x + y^2 \); (f) \( \varphi(x,y) = x - y^2 \).

Solution. Note that \( \int_{-\pi}^{\pi} \sin \theta \, d\theta = 0 \), \( \int_{-\pi}^{\pi} \cos \theta \, d\theta = 0 \), \( \int_{-\pi}^{\pi} \cos^2 \theta \, d\theta = \pi \), and \( \int_{-\pi}^{\pi} \sin^2 \theta \, d\theta = \pi \). For all \( a = (\alpha, \beta) \in \mathbb{C} \) and any \( r > 0 \) we have
(a)
\[
\varphi(a + re^{i\theta}) = \varphi(\alpha + r \cos \theta, \beta + r \sin \theta) = (\alpha + r \cos \theta)^2 + (\beta + r \sin \theta)^2
\]
\[
= \alpha^2 + 2\alpha r \cos \theta + r^2 \cos^2 \theta + \beta^2 + 2\beta r \sin \theta + r^2 \sin^2 \theta
\]
\[
= \alpha^2 + \beta^2 + 2\alpha r \cos \theta + 2\beta r \sin \theta + r^2.
\]
Thus
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta}) \, d\theta = \frac{\alpha^2 + \beta^2 + \alpha r \pi - \beta r \pi}{2\pi} \int_{-\pi}^{\pi} \cos \theta \, d\theta + \frac{\beta r \pi - \beta r \pi}{2\pi} \int_{-\pi}^{\pi} \sin \theta \, d\theta + r^2
\]
\[
= \alpha^2 + \beta^2 + r^2 \geq \alpha^2 + \beta^2 = \varphi(a).
\]
Thus
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta}) \, d\theta = \alpha + \beta^2 + \frac{r}{2\pi} \int_{-\pi}^{\pi} \cos \theta \, d\theta + \frac{\beta r}{\pi} \int_{-\pi}^{\pi} \sin \theta \, d\theta + \frac{r^2}{2\pi} \int_{-\pi}^{\pi} \sin^2 \theta \, d\theta \]
\[ = \alpha + \beta^2 + \frac{r^2}{2} \geq \alpha + \beta^2 = \varphi(a). \]
Hence, \( \varphi \in \text{Subhar}(G) \).

Thus
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta}) \, d\theta = \alpha - \beta^2 + \frac{r}{2\pi} \int_{-\pi}^{\pi} \cos \theta \, d\theta - \frac{\beta r}{\pi} \int_{-\pi}^{\pi} \sin \theta \, d\theta - \frac{r^2}{2\pi} \int_{-\pi}^{\pi} \sin^2 \theta \, d\theta \]
\[ = \alpha - \beta^2 - \frac{r^2}{2} \leq \alpha - \beta^2 = \varphi(a). \]
Hence, \( \varphi \in \text{Superhar}(G) \).

Exercise 2. Let \( \text{Subhar}(G) \) and \( \text{Superhar}(G) \) denote, respectively, the sets of subharmonic and superharmonic functions on \( G \).
(a) Show that \( \text{Subhar}(G) \) and \( \text{Superhar}(G) \) are closed subsets of \( C(G, \mathbb{R}) \).
(b) Are versions of Harnack’s Theorem valid for subharmonic and superharmonic functions?

Solution. Not available.

Exercise 3. (This exercise is difficult.) If \( G \) is a region and if \( f : \partial \omega G \to \mathbb{R} \) is a continuous function let \( u_f \) be the Perron Function associated with \( f \). This defines a map \( T : (\partial \omega G, \mathbb{R}) \to \text{Har}(G) \) by \( T(f) = u_f \).
Prove:
(a) \( T \) is linear (i.e., \( T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \)).
(b) \( T \) is positive (i.e., if \( f(a) \geq 0 \) for all \( a \) in \( \partial \omega G \) then \( T(f)(z) \geq 0 \) for all \( z \) in \( G \)).
(c) \( T \) is continuous. Moreover, if \( \{f_n\} \) is a sequence in \( C(\partial \omega G, \mathbb{R}) \) such that \( f_n \to f \) uniformly then \( T(f_n) \to T(f) \) uniformly on \( G \).
(d) If the Dirichlet Problem can be solved for \( G \) then \( T \) is one-one. Is the converse true?

Solution. Not available.

Exercise 4. In the hypothesis of Theorem 3.11, suppose only that \( f \) is a bounded function on \( \partial \omega G \); prove that the conclusion remains valid. (This is useful if \( G \) is an unbounded region and \( g \) is a bounded continuous function on \( \partial G \). If we define \( f : \partial \omega G \to \mathbb{R} \) by \( f(z) = g(z) \) for \( z \) in \( \partial G \) and \( f(\infty) = 0 \) then the conclusion of Theorem 3.11 remains valid. Of course there is no reason to expect that the harmonic function will have predictable behavior near \( \infty \) — we could have assigned any value to \( f(\infty) \). However, the behavior near points of \( \partial G \) can be studied with hope of success.)

Solution. Not available.

Exercise 5. Show that the requirement that \( G_1 \) is bounded in Corollary 3.5 is necessary.
11.3 Hadamard Factorization Theorem

Exercise 1. Let \( f \) be analytic in a region \( G \) and suppose that \( f \) is not identically zero. Let \( G_0 = G - \{z : f(z) = 0\} \) and define \( h : G_0 \to \mathbb{R} \) by \( h(z) = \log |f(z)| \). Show that \( \frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y} = \frac{f'}{f} \) on \( G_0 \).

Solution. Let \( f \) be analytic in a region \( G \) and suppose that \( f \) is not identically zero. Let \( G_0 = G - \{z : f(z) = 0\} \), then \( h : G_0 \to \mathbb{R} \) given by \( h(z) = \log |f(z)| \) is well defined as well as \( \frac{f'}{f} \) is well defined on \( G_0 \).

Let \( f = u(x, y) + iv(x, y) = u + iv \). Since \( f \) is analytic, the Cauchy-Riemann (C-R) equations \( u_x = v_y \) and \( u_y = -v_x \) are satisfied. We have by p. 41 Equation 2.22 and 2.23

\[
\frac{f''}{f} = u_x + iv_x \quad \text{and} \quad \frac{f'}{f} = -iu_y + v_y
\]

and thus

\[
2f' = u_x + iv_x - iu_y + v_y
\]

implies

\[
f' = \frac{1}{2} \left( u_x + iv_x - iu_y + v_y \right) = \frac{1}{2} \left( u_x - iu_y - iu_y + u_x \right) = \frac{1}{2} \left( 2u_x - 2iu_y \right) = u_x - iu_y.
\]

Therefore,

\[
f' = u_x + iu_y = \frac{u_x}{u^2 + v^2} + iu_y = u + iv.
\]

(11.6)

Next, we calculate \( \frac{\partial h}{\partial x} = h_x \) and \( \frac{\partial h}{\partial y} = h_y \), where \( h(z) = \log |f(z)| = \frac{1}{2} \log(u^2 + v^2) \). Using the chain rule, we get

\[
h_x = h_x u_x + h_y v_x = \frac{u}{u^2 + v^2} u_x + \frac{v}{u^2 + v^2} v_x
\]

and

\[
h_y = h_x u_y + h_y v_y = \frac{u}{u^2 + v^2} u_y + \frac{v}{u^2 + v^2} v_y
\]

and hence

\[
\frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y} = h_x - ih_y
\]

\[
= \frac{u}{u^2 + v^2} u_x + \frac{v}{u^2 + v^2} v_x - i \frac{u}{u^2 + v^2} u_y - i \frac{v}{u^2 + v^2} v_y
\]

\[
= \frac{u}{u^2 + v^2} (u_x - iu_y) + \frac{v}{u^2 + v^2} (v_x - iv_y)
\]

\[
= \frac{u}{u^2 + v^2} (u_x - iu_y) - \frac{iv}{u^2 + v^2} (u_x - iu_y)
\]

\[
= \frac{u - iv}{u^2 + v^2} (u_x - iu_y)
\]

\[
= \frac{u - iv}{u^2 + v^2} (u_x - iu_y)
\]

\[
= \frac{u - iv}{u + iv}.
\]