\[ \begin{align*}
\text{Polar form of a complex number:} & \\
Z &= x + iy, \quad r = r \cos \theta, \quad y = r \sin \theta \\
&\Rightarrow \quad r^2 = x^2 + y^2 \\
&\Rightarrow \quad r = \sqrt{x^2 + y^2} \\
&\Rightarrow \quad |Z| = \sqrt{x^2 + y^2} = r \\
Z &= x + iy = r \cos \theta + i r \sin \theta \\
&\Rightarrow \quad Z = r (\cos \theta + i \sin \theta) \\
&\Rightarrow \quad Z = re^{i\theta} \\
\theta &= \tan^{-1} \left( \frac{y}{x} \right)
\end{align*} \]

\text{Properties of the modulus:} \\
If \ Z = x + iy \ (x, y \ \text{real}), \ then \ |Z| = \sqrt{x^2 + y^2}.

(a) \ |Z| = |\bar{Z}| \\
(b) \ |z_1 z_2| = |z_1| |z_2| \\
(c) \ |\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \\
(d) \ \text{the distance between two points} \ Z_1, Z_2 \ \text{is} \\
|Z_1 - Z_2| = |Z_2 - Z_1|
Now

\[ f(z+\Delta z) = f(x+\Delta x, y+\Delta y) \]

\[ = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) \]

\[ = [u(x, y) + \Delta x \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \frac{1}{2} \Delta x^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ + i \left[ u(x, y) + (\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}) + \frac{1}{2} \right] ] \]

\[ = [u(x, y) + i u(x, y)] + \Delta x \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \Delta y \left( \frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x} \right) \]

\[ - \text{ neglecting higher order derivative...} \]

\[ = f(z) + \Delta x \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \Delta y \left( \frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x} \right) \]

\[ \Rightarrow \frac{f(z+\Delta z) - f(z)}{\Delta z} = \Delta x \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \Delta y \left( \frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x} \right) \]

\[ = \Delta x \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) \cdot \left( \frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x} \right) = \Delta z \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) \]

\[ \Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \]

\[ f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \]

Similarly, \( f'(z) = \frac{\partial v}{\partial y} \)

and also given \( u_x = -1 \), \( u_y = 1 \) and these are continuous in \( D \).

\[ \therefore f(z) \text{ is analytic in } D. \]
(ii) To show that \( f'(z) \) does not exist.

Now
\[
f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - f(0)}{z}
\]

\[
f'(z) = \lim_{z \to 0} \frac{e^{z^2} - 1}{z}
\]

(a) along x-axis: then \( y = 0 \) in (A) we get
\[
f'(x) = \lim_{x \to 0} \frac{x^3 + i(2x)}{x^3} = 1 + i
\]

(b) along y-axis: then \( x = 0 \) in (B) we get
\[
f'(y) = \lim_{y \to 0} \frac{-y^3 + i(y^3)}{y^3} = i - 1 = 1 - i
\]

(c) along straight line \( y = x \) in (A)
\[
f'(0) = \lim_{x \to 0} \frac{(x^2 - x^2) + i(x^3 + x^3)}{(x^2 - x^2) + i(x^3 + x^3)} = \lim_{x \to 0} \frac{x^3}{x^3} = 1
\]

The value of \( f'(z) \) is not constant in any region of different paths.

Thus \( f(z) \) is not analytic at \( z = 0 \).

Problem: If \( u - v = (x - y)(x^2 + 4y + 6y^2) \) and \( f(z) = u + iv \) is an analytic function of \( z = x + iy \), find \( f(z) \) in terms of \( z \).

Solution:

Given
\[
f(z) = u + iv
\]

\[
f(z) = \text{iu} - v
\]

Adding (1) and (2), we get
\[
(1 + i)f(z) = (u-v) + i(u+v)
\]

\[
f(z) = u + iv
\]