1 Board work

1.1 Tuesday, 29 September 2015

1. For a cluster point every deleted neighborhood actually has infinitely many points of $A$.

   Example. Let $0$ be a cluster point of $A$. Then there must be a point $a \in A$ such that $a \in D_1(0)$. Since $D_1|a|(0)$ is also a deleted neighborhood of $0$ there must be a point $b \in A$ such that $b \in D_1|a|(0)$. Next there must be a point $c \in A$ with $c \in D_1|b|(0)$. And so on.

2. Sequence. Think of a sequence as a collection of a first term, a second term,..., an $n^{th}$ term,..., ad infinitum.

3. Tail. A tail of a sequence is new sequence obtained by removing finitely many initial terms of the given sequence. Thus $x_N, x_{N+1}, x_{N+2}, \ldots$ is a tail of the sequence $x_1, x_2, x_3, \ldots$


   $x_n$ converges to $x$ if and only if for every $\epsilon > 0$ the number of terms of $(x_n)$ outside $B_\epsilon(x)$ is finite. (In which case the number of terms of $(x_n)$ inside $B_\epsilon(x)$ will automatically be infinity).

5. Note that (i) $\iff$ (ii) $\iff$ (iii) where
(i) \( (x_n) \) has finitely many terms outside \( B_\epsilon(x) \),
(ii) \( (x_n) \) has a tail with zero terms outside \( B_\epsilon(x) \), and
(iii) \( (x_n) \) has a tail with all terms inside \( B_\epsilon(x) \).

It follows that \( (x_n) \) converges to a limit \( l \) if and only if given any \( \epsilon > 0 \) there is a tail of \( (x_n) \) with all terms of the tail inside \( B_\epsilon(l) \).

6. It follows that if for some \( \epsilon > 0 \) the number of terms of \( (x_n) \) outside \( B_\epsilon(x) \) is infinity then \( x_n \) cannot converge to \( x \).

7. Every convergent sequence is bounded. Assume \( x_n \to l \). Then a tail of \( (x_n) \) is entirely contained in \( B_1(l) \). So every element of the tail is bounded by \(|l| + 1\). What about the terms of the sequence that precede the tail? Are they bounded also? There are only finitely many terms in the sequence \( (x_n) \) which do not belong to the tail. Since the maximum of finitely many finite numbers is also finite, we get a finite bound for the entire sequence.

1.2 Wednesday, 30 September 2015

1. Sandwich theorem. \( a_n \leq b_n \leq c_n \) with \( a_n \to a \) and \( c_n \to l \). Then \( b_n \to l \).

Since \( a_n \to l \), there exists \( n_a \) such that \( l - \epsilon < a_n \) for all \( n > n_a \). Since \( c_n \to l \), there exists \( n_c \) such that \( c_n < l + \epsilon \) for all \( n > n_c \). Therefore \( b_n \in B_\epsilon(l) \) whenever \( n \) greater than both \( n_a \) and \( n_b \). (In other words, every \( B_\epsilon(l) \) contains a tail of \( (b_n) \).)

2. If \( a_n \to a \) the \( ca_n \to ca \) for every \( c \in \mathbb{R} \).

Trivial for \( c = 0 \).
For \( c \neq 0 \) there exists \( n_0 \) such that \( a_n \in B_{\epsilon/|c|}(a) \) for all \( n > n_0 \). So \( ca_n \in B_\epsilon(a) \) for all \( n > n_0 \). (In other words, an arbitrary neighbourhood of \( ca \) and contains a tail of the sequence \( (ca_n) \).)

3. \( a_n \to a \) and \( b_n \to b \) implies that \( a_n + b_n \to a + b \).

Given arbitrary \( \epsilon > 0 \) there exists \( n_a \) such that
\[
|a - a_n| < \epsilon/2 \text{ for all } n > n_a.
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