DEFINING ADDITION + MULTIPLICATION

• Let \( f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \),
  \( g(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m \).
  
  \[ f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n \]
  \[ f(x) \cdot g(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1) x + \ldots + a_n b_{n-1} x \]

• **THEOREM:** The set \( R[x] \) of all polynomials with coefficients in a ring \( R \) is a ring under polynomial addition and multiplication.

• **SOME FEATURES:**
  
  • If \( R \) is commutative, so is \( R[x] \).
  
  • If \( R \) has unity \( 1 \neq 0 \), then \( 1 \) is also unity for \( R[x] \).

• **EXAMPLE:** Consider \( \mathbb{Z}_2[x] \).
  
  \[ (x+1)^2 = (x+1)(x+1) = x^2 + (1+1)x + 1 = x^2 + 1 \]
  \[ (x+1) + (x+1) = (1+1)x + (1+1) = 0x + 0 = 0 \]

→ TWO INDETERMINATES

• \( (R[x])[y] \) is a ring of polynomials in \( y \) with coefficients that are polynomials in \( x \).

• \( (R[x])[y] \cong (R[y])[x] \)

• **NOTATION:** \( (R[x])[y] = R[x, y] \)

→ RANDOM NOTES

• If \( D \) is an integral domain, so is \( D[x] \).

• If \( D \) is a field, \( D[x] \) is an integral domain.