Figure 4.8: Bounded region in $\mathbb{R}^2$, enclosed in a rectangle.

**Definition 330** If $\iint_{\mathcal{R}} F(x, y) \, dA$ exists, then we define

$$\iint_{\mathcal{D}} f(x, y) \, dA = \iint_{\mathcal{R}} F(x, y) \, dA$$

This definition makes sense since, as we can see in figures 4.9 and 4.10, the portion of the graph of $F$ which is not 0 is identical to the graph of $f$. The portion which is 0 will not contribute to the integral. In particular, this means that for our definition, it does not matter which rectangle $\mathcal{R}$ we select. In the case, $f(x, y) \geq 0$, $\iint_{\mathcal{D}} f(x, y) \, dA$ corresponds to the volume of the solid which lies above $\mathcal{D}$ and below the graph of $z = f(x, y)$.

We still must be able to compute $\iint_{\mathcal{R}} F(x, y) \, dA$. This is not always a simple task. But it is for certain regions, which we consider next.

### 4.3.2 Regions of Type I

When describing a region, one has to give the condition $x$ and $y$ must satisfy so that a point $(x, y)$ lies in the region. A region is said to be of type I if $x$ is between two constants, and $y$ is between two continuous functions of $x$. More precisely, we have the following definition:

**Definition 331** A plane region $\mathcal{D}$ is said to be of **type I** if it is of the form

$$\mathcal{D} = \{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x) \}$$
Figure 4.15: Region between $y = x$ and $y = x^2$
Method 1  Treating $\Omega$ as a type I region.

\[
\iiint_{\Omega} (\sqrt{x} - y^2) \, dA = \int_0^1 \int_{x^2}^1 (\sqrt{x} - y^2) \, dy \, dx
\]

\[
= \int_0^1 \left[ y\sqrt{x} - \frac{y^3}{3} \right]_{x^2}^1 \, dx
\]

\[
= \int_0^1 \left( \frac{x^4}{4} - \frac{x^2}{2} - \frac{x^6}{3} \right) \, dx
\]

\[
= \int_0^1 \left( \frac{2}{3}x^2 - x^2 + \frac{x^6}{3} \right) \, dx
\]

\[
= \left( \frac{8}{21}x^7 - \frac{2}{7}x^7 + \frac{x^{12}}{21} \right) \bigg|_0^1
\]

\[
= \frac{8}{21} - \frac{2}{7} + \frac{1}{21}
\]

\[
= \frac{1}{7}
\]

Method 2  Treating $\Omega$ as a type II region.

\[
\iiint_{\Omega} (\sqrt{x} - y^2) \, dA = \int_0^1 \int_0^{x^2} (\sqrt{x} - y^2) \, dy \, dx
\]

\[
= \int_0^1 \left[ \frac{2}{3}x^3 - xy^2 \right]_{y^2}^{x^2} \, dy
\]

\[
= \int_0^1 \left( \frac{2}{3}x^4 - \frac{2}{3}x^2 - \frac{2}{3}y^6 + y^6 \right) \, dy
\]

\[
= \int_0^1 \left( \frac{2}{3}y^3 - y^2 + \frac{y^6}{3} \right) \, dy
\]

\[
= \left( \frac{8}{21}y^7 - \frac{2}{7}y^7 - \frac{y^{12}}{21} \right) \bigg|_0^1
\]

\[
= \left( \frac{8}{21} - \frac{2}{7} + \frac{1}{21} \right)
\]

\[
= \frac{1}{7}
\]

Example 349  Evaluate $\iiint_{\Omega} \cos \frac{xy^2}{2} \, dA$ where $D$ is the region bounded by $x = 1$, $y = 0$ and $y = x$. 

\[
\int_0^1 \int_0^{x^2} \int_0^{x^2} \cos \frac{xy^2}{2} \, dx \, dy \, dz
\]

\[
= \int_0^1 \int_0^{x^2} \cos \left( \frac{y^2}{2} \right) \, dy \, dx
\]

\[
= \int_0^1 \left[ \frac{2}{3} \sin \left( \frac{y^2}{2} \right) \right]_{y^2}^{x^2} \, dx
\]

\[
= \int_0^1 \left( \frac{2}{3} \sin \left( \frac{x^4}{2} \right) - \frac{2}{3} \sin \left( \frac{y^4}{2} \right) \right) \, dx
\]

\[
= \left( \frac{8}{21} \sin \left( \frac{x^7}{2} \right) - \frac{2}{7} \sin \left( \frac{x^{12}}{21} \right) - \frac{1}{21} \right) \bigg|_0^1
\]

\[
= \frac{8}{21} - \frac{2}{7} + \frac{1}{21}
\]

\[
= \frac{1}{7}
\]