\[
\lim_{x \to -4} \sqrt{x} = \text{DNE (Not real)}
\]

\[
\lim_{x \to 0} \sqrt{x} = \text{(DNE) (no left side)}
\]

\[
\lim_{x \to 2} \frac{x^2 - 4}{x + 1} = \frac{2^2 - 4}{2 + 1} = \frac{0}{3} = \frac{2}{3}
\]

\[
\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{3^2 - 9}{3 - 3} = \frac{0}{0} = \text{Undetermined indeterminate}
\]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>1</td>
</tr>
<tr>
<td>3.001</td>
<td>1</td>
</tr>
<tr>
<td>2.9</td>
<td>2</td>
</tr>
<tr>
<td>2.99</td>
<td>2</td>
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<tr>
<td>2.999</td>
<td>2</td>
</tr>
<tr>
<td>2.9999</td>
<td>2</td>
</tr>
</tbody>
</table>

When you have indeterminate - find rational expression:

\[
\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = \frac{x+3}{1}
\]

\[
\lim_{x \to 3} x + 3 = 6
\]

very close to 6
Piecewise Functions

can be solved either graphically or algebraically.

\[ P(x) = \begin{cases} 2 + x & x \leq 0 \\ 2 - x & x > 0 \end{cases} \]

<table>
<thead>
<tr>
<th>x</th>
<th>y = 2 + x</th>
<th>y = 2 - x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} f(x) = 2 \]

\[ f(0) = 2 + 0 = 2 \]

cont. at \( x = 0 \)
Last Day of Limits!

The rate of change of one quantity with respect to another is mathematically equivalent to the slope of the tangent line.

\[ f'(x) \]

\[ f(x) \] is # of social security beneficiaries in yr \( t \)
\[ t=0 \rightarrow 1990 \] (in millions)

In year 2000 \( t=10 \)
\[ f(x) = \frac{10}{30} = .5 \]

\[ \frac{10}{30} = 0.333 \]

\[ \frac{1.5}{10} = 0.15 \]

\[ = 1,500,000 \]

Average rate of change: of \( f(x) \) over the interval

\[ \frac{f(x+h) - f(x)}{h} \]

Instantaneous rate of change: of \( f(x) \) with respect to \( x \) is:

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Derivative of a function is defined: new function Domain is

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

1st derivative

\[ k \text{ or } \frac{dy}{dx} \text{ (integration) } = Dx(x,y) \]
\[ f(x) = 3/2x^{3/2} \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(2x+2h)^{3/2} - 3(2x+2h)^{3/2}}{h(2x+2h)(2x)} \]

\[ = \lim_{h \to 0} \frac{3}{h} \cdot \frac{2x+2h}{2x+2h} \cdot \frac{2x}{2x+2h} \cdot \frac{1}{2x} \cdot \frac{3}{2} \]

\[ = -\frac{3}{4x^{1/2}} \]

Where is the tangent line horizontal? No where.

Find equation of tangent line at \( x = 1 \)

\[ y_1 : f(1) = 3/2(1) = 3/2 \]

\[ f'(1) = -\frac{3}{4}(1)^{1/2} \]

\[ m = -\frac{3}{2} \]

\[ y - \frac{3}{2} = -\frac{3}{2} (x-1) \]

What is the rate of change? (slope) \(-\frac{3}{2}\)
Chain Rule for Power Functions

\[ y = [f(x)]^n \quad p(x) \neq \text{plain } x \]

\[ y' = n [f(x)]^{n-1} \cdot f'(x) \]

EX.

\[ f(x) = (x^4 - 5)^5 \]

\[ f'(x) = 5 (x^4 - 5)^4 \cdot 4x^3 \]

\[ f''(x) = 20x^3 (x^4 - 5)^4 \]

\[ f(x) = (5x+3)^3 \]

\[ f'(x) = 3 (5x+3)^2 \cdot 5 \]

\[ f''(x) = 15 (5x+3)^2 \]

\[ f(x) = \sqrt[3]{x-4} \]

\[ f'(x) = \frac{1}{3} (4-x)^{-\frac{2}{3}} \]

\[ f''(x) = -\frac{2}{9} (4-x)^{-\frac{5}{3}} \]

\[ f(x) = (2x+1)^2 (x^2-6x) \]

\[ f'(x) = 2 (2x+1)^2 (2x) + 2 (2x+1) (2) \cdot (x^2-6x) \]

\[ = 2 (2x+1) [2x(x^2-6x) + 2 (x^2-6x)] \]

\[ = 2(2x+1) (4x^2 + x - 12) \]
\[ P(x) = b^x \quad p(x) = e^x \]

\[ \begin{array}{c}
(0,1) \\
D = (-\infty, \infty) \\
R = (0, \infty)
\end{array} \]

\[ p(x) = e^x \\
p'(x) = e^x \quad p''(x) = 5e^x \]

\[ p(x) = 4e^x + 8x^2 + 7x - 14 \\
p'(x) = 4e^x + 16x + 7 \]

\[ p(x) = e^x \frac{e^x}{x^2} \quad p''(x) = e^x \left( x + 2x \right) \text{ or } e^x(x+2) \]

\[ p(x) = \frac{1-e^x}{1+e^x} \]

\[ p'(x) = (1+e^x)(e^x) - (1-e^x)(e^x) \]

\[ = -e^x - e^{2x} - e^x + e^{2x} \]

\[ = -2e^x \]

\[ p''(x) = \frac{-2e^x}{(1+e^x)^2} \]
Averages

 Avg cost, rev, profit:

\[ C(x) = \frac{\text{Fixed costs}}{x} \quad R(x) = \frac{\text{Revenue}}{x} \quad \bar{P}(x) = \frac{\text{Profit}}{x} \]

Marginal:

\( \bar{C}(x) \quad R'(x) \quad P'(x) \)

\[ C(x) = 150,000 + 30x \]

avg. cost + marginal cost function eval. x = 1000

\[ C'(x) = \frac{150,000 + 30x}{x} \]

\[ C'(1000) = 81.80 \]

avg. fixed cost 8180

\[ C'(x) = 150,000 - 30 \quad C(1000) = 8180 \]

\[ \bar{R}(x) = 300x - \frac{1}{2}30x^2 \rightarrow \text{avg. rev + marginal } x = 1000 \]

\[ \bar{R}(x) = \frac{300x - \frac{1}{2}30x^2}{x} \]

\[ \bar{R}(1000) = 184.66 \quad \text{C7} \]

\[ R''(x) = -\frac{1}{2}30 \]

\[ R''(1000) = -0.03 \]

Average revenue is decreasing by 0.03

When 1001 are produced the average revenue is £206.04
Cobb-Douglas Productivity Function

\[ P(x, y) = ax^b y^{1-b} \]

Economists use this to determine the number of units produced from the utilization of units of labor and units of capital.

\( x \): The productivity of a steel manufacturing company is approximated by:

\[ P(x, y) \cdot 10^x^a y^b^8 \]

\( \xi \): If the company currently uses 1000 units of labor and 2000 units of capital, how many units of steel produced:

\[ P(1000, 2000) \cdot 10(1000^\theta 2000^8) \]

201.01 \& 17411 units of steel

Partial Derivatives

When we take the derivative with respect to 1 of the variables and hold the other variable constant, we obtain partial:

\( P_x(x, y) \rightarrow \) we take derivative of \( x \)s and treat \( y \)s as constants

\( P_y(x, y) \rightarrow \) we take derivative of \( y \)s and treat \( x \)s as constants

\( \xi \): \( P(x, y) = 3x^2y - 4y \)

\( P_x(x, y) = 5(2x)y - 0 \)

\( = 6xy \)

\( = \) 

\( P_y(x, y) = 3x^2(1) - 4(1) \)

\( = 3x^2 - 4 \)
Vertical Asymptotes: Another place where limit DNE is where a function has $VA = \text{really large (} \pm \infty \text{)}$ or really small (-\infty) as $x \to a$.

\[ f(x) = \frac{1}{x}, \quad x \neq 0 \]

\[ f(x) \]

\[ \lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{VA: } x = 0 \]

\[ \lim_{x \to 0^+} \frac{1}{x} = +\infty \]

\[ p(x) = x^2 \]

\[ \lim_{x \to 0^-} \frac{1}{x^2} = +\infty \quad \text{VA: } x = 0 \]

\[ \lim_{x \to 0^+} \frac{1}{x^2} = +\infty \]

\[ p(x) = \frac{x-3}{x^2-4x+3} = \frac{x-3}{(x-3)(x-1)} = \frac{1}{x-1}, \quad x \neq 1 \]

9. $\lim_{x \to 1^-} \frac{1}{x-1} = -\infty \quad \text{VA: } x = 1$

10. $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$
Anti-Derivative

A function $F$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

$f^3(x) = x^4$

$f(x) = \frac{1}{5}x^5$

3 family of antiderivatives of $f^3(x) = x^4$

If $g(x)$ is the antiderivative of function $f(x)$, every antiderivative $F(x)$ must be of the form $F(x) = g(x) + c$

The process of finding the antiderivative is called antidifferentiation or integration

\[
\int f(x)dx = F(x) + c
\]

Integrate with respect to $x$

\[
\int x^4 dx = x^5 + c
\]

Rules:

Constant rule: $\int kdx = kx + c$

$\int dx = x + c$
Power Rule: \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \)

\[
\int x^3 \, dx = \frac{x^4}{4} + C = \frac{1}{4}x^4 + C
\]

\[
\int \frac{1}{x^2} \, dx = \frac{x^{-1}}{-1} + C = -x^{-1} + C
\]

Constant multiplier rule: \( \int k \cdot x^n \, dx = k \cdot \frac{x^{n+1}}{n+1} + C \)

\[
\int 3x^4 \, dx = 3 \cdot \frac{x^5}{5} + C = \frac{3}{5}x^5 + C
\]

\[
\int 4x^3 \, dx = \frac{4x^4}{4} + C = x^4 + C
\]

\[
\int (k(x)^i \cdot q(x)^i \ldots) \, dx = \int k(x) \, dx \cdot \int q(x) \, dx \cdot \ldots
\]

\[
\int (2x^5 - 3x^3 + 1) \, dx = 2 \cdot \frac{x^6}{6} - 3 \cdot \frac{x^4}{4} + lx + C
= \frac{1}{3}x^6 - x^4 + x + C
\]

\[
\int \left( \frac{3}{x^4} - \frac{2}{x^4} \right) \, dx = \int \left( 2x^{2/3} - 3x^{-4} \right) \, dx
= \frac{2}{3}x^{5/3} + x^{-3} + C
\]

\[
\int \frac{x^4 - 8x^3}{x^2} \, dx = \int (x^2 - 8x) \, dx
= \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + C = \frac{1}{3}x^3 - 4x^2 + C
\]

\[
\int \left( \frac{3}{3(\sqrt[3]{x})} - \frac{1}{x} \right) \, dx = \int \left( 8x^{1/3} - C \cdot x^{-1/3} \right) \, dx
= \frac{2}{3}x^{4/3} - \frac{4}{3}C \cdot x^{1/3} + C = \frac{2}{3}x^{4/3} - \frac{4}{3}C \cdot x^{1/3} + C
\]
U-Substitution

Reverse the Chain Rule
\[ S \left[ (p(x))^{\frac{1}{n}} \right] \cdot p'(x) \, dx = \frac{[p(x)]^{\frac{1}{n}}}{\frac{1}{n+1}} + c \]

\[ S e^{p(x)} \cdot p'(x) \, dx = e^{p(x)} + c \]

\[ S \left[ \frac{p(x)}{p'(x)} \right] \cdot e^{p(x)} \, dx = \ln |p(x)| + c \]

ex.

1. \[ S \left( 2x^3 - 3 \right)^{\frac{1}{2}} \, dx = \frac{1}{3} \left( 2x^3 - 3 \right)^{\frac{3}{2}} + c \]
   \[ du = 6x^2 \, dx \]

2. \[ S e^{5x} \, dx = \frac{1}{5} e^{5x} + c \]
   \[ u = 5x \]
   \[ du = 5 \, dx \]

3. \[ S \frac{2x}{x^2 + 4} \, dx = \ln |1 + x^2| + c \]
   \[ u = 4 + x^2 \]
   \[ du = 2x \, dx \]
4. \[ \int \frac{(x^2 + 2x + 5)^5}{x}(x+1)\,dx \]

\[ u = x^2 + 2x + 5 \]
\[ du = 2x + 2\,dx \]
\[ \frac{1}{2}du = (x+1)\,dx \]
\[ \int u^5 \cdot \frac{1}{2}du = \frac{1}{2} \int u^5 \,du \]
\[ = \frac{1}{2} \cdot \frac{u^6}{6} + C \]
\[ = \frac{1}{12}(x^2 + 2x + 5)^6 + C \]

5. \[ \int (4x - 6)(x^2 - 3x + 7)^{-4}\,dx \]

\[ u = x^2 - 3x + 7 \]
\[ du = 2x - 3\,dx \]
\[ \frac{1}{2}du = (4x - 6)\,dx \]
\[ \int u^{-4} \cdot \frac{1}{2}du = \frac{1}{2} \int u^{-4} \,du \]
\[ = \frac{1}{2} \cdot \frac{1}{-3} u^{-3} + C \]
\[ = \frac{1}{-3} (x^2 - 3x + 7)^{-3} + C \]

6. \[ \int e^{3x} \,dx \]

\[ u = -3x \]
\[ du = -3\,dx \]
\[ \int e^u \cdot -\frac{1}{3} du = -\frac{1}{3} e^u + C \]
\[ = -\frac{1}{3} e^{-3x} + C \]

7. \[ \int \frac{x}{x^2 - 9} \,dx \]

\[ u = x^2 - 9 \]
\[ du = 2x\,dx \]
\[ \frac{1}{2}du = x\,dx \]
\[ \int u^{-1} \cdot \frac{1}{2}du = \frac{1}{2} \int u^{-1} \,du \]
\[ = \frac{1}{2} \ln |u| + C \]
\[ = \frac{1}{2} \ln |x^2 - 9| + C \]
1. cont. b. \[ \int x(x+4)^3 \, dx = \int (x+4)^3 \, du \]

\[ u = x+4 \Rightarrow u-4 = x \]

\[ du = dx \]

\[ \frac{1}{5} (x+4)^5 - (x+4)^4 + C = \frac{u^5}{5} - 4 \frac{u^4}{4} + C \]

\# 5

2. a. \[ f''(x) = \frac{-10}{x^2} \quad f(1) = 20 \]

\[ f(x) = \int -10x^{-2} \, dx = 10x^{-1} + 10 \]

\[ f'(x) = 10x^{-1} \]

\[ f(x) = \frac{1}{x} + 10 \]

b. \[ f''(x) = x^3 + 1 \quad f(2) = 5 \]

\[ f(x) = \int x^3 + x \, dx = \frac{x^4}{4} + \frac{x^2}{2} + C \]

\[ 5 = \frac{4}{4} + \frac{4}{2} + C \]

\[ 5 = 6 + C \]

\[ -1 = C \]

\#16 \#7 (1u)

3. a. \[ \int_1^2 (5 - 10x^3) \, dx \]

\[ = \left[ 5x - \frac{10x^2}{2} \right]_1^2 = \left[ 5(2) - \frac{10(2)^2}{2} \right] - \left[ 5(1) - \frac{10(1)^2}{2} \right] \]

\[ = -1 \]
4. \[ V(t) = 500(t-12) \quad 0 \leq t \leq 10 \]

a. Find total loss in value during the first 5 yr

\[ V(t) = \int_0^5 500(t-12) \, dt = \frac{500}{2} \cdot \frac{5}{2} = 625 \]

b. Find average loss over the first 5 yr

\[ \frac{1}{5-0} \int_0^5 500(t-12) \, dt = 125 \]

5. a. \[ y = x^3 - 1 \]

\[ y = x - 2 \]

\[ -2 \leq x \leq 1 \]

\[ \int_{-2}^{1} [(x^3 - 1) - (x - 2)] \, dx = 7.5 \]

b. \[ y = \frac{2x}{x^2 + 1} \]

\[ y = 0 \]

\[ -2 \leq x \leq 2 \]

\[ \int_{-2}^{2} [0 - \frac{2x}{x^2 + 1}] \, dx + \int_{0}^{2} [0 - \frac{2x}{x^2 + 1}] \, dx = 1.38 \]
You wish to have £200,000 for a down payment on a home, but you only have £15,000. You invest the £15,000 at 10% compounded quarterly. How long until you have £200,000?

\[ A = P(1 + C)^n \]
\[ C = \frac{r}{m} \]
\[ n = m \cdot t \]

\[ 200000 = 15000 \left(1 + \frac{10}{4}\right)^{4t} \]

\[ \ln(1.3333) = 0.02546 \]
\[ \ln(1.3333) = 4 \cdot \ln(1.025) \]
\[ \frac{\ln(1.3333)}{4} = \frac{\ln(1.025)}{4} \]
\[ \ln(1.3333)/4 \Rightarrow \ln(1.025) \]

**Effective Rate (annual percentage yield)**

APY

Simple interest rate that produces same results as compounded rate in 1 yr

\[ R_{eff} = (1 + \frac{r}{m})^m - 1 \]

**Find the effective rate**

1. 10% compounded monthly

\[ R_{eff} = (1 + \frac{0.10}{12})^{12} - 1 \]

\[ = 0.1047 \]

\[ 10.47\% \]

2. 9% compounded continuously

\[ R_{eff} = e^{0.09} - 1 \]

\[ = 0.42\% \]
Present value: How much should you deposit now in order to withdraw an amount each period for a set # of yrs.

\[ PV = PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right] \]

How much should you deposit now in order to withdraw $1000 each month for 2 yrs at 12% comp. cont.?

\[ N = 24 \]
\[ i% = 12 \]
\[ a \] PV = 0
\[ PMT = 1000 \]
\[ FV = 0 \]
\[ c/y = 12 \]
\[ i/y = 10 \]

End

Amortization: When a loan is repaid in regular installments for a set # of yrs. use PV formula.

Ex. A family pays a $180,000 condo. They pay 10% and finance at 4.5% comp. monthly for 30 yrs.

What is the monthly payment?
Down: $18,000
PV: $162,000
\[ I = 820.83 (360) - 162,000 = 233,479 \]