The Absolute Value Function, and its Properties

One of the most used functions in mathematics is the absolute value function. Its definition and some of its properties are given below.

**Absolute Value Function** The absolute value of a real number \(x\), \(|x|\), is

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

The graph of the absolute value function is shown below

![Graph of the absolute value function](image)

**Example 1**

\[
|2| = 2, \quad |-2| = 2
\]

The absolute value function is used to measure the distance between two numbers. Thus, the distance between \(x\) and 0 is \(|x - 0| = |x|\), and the distance between \(x\) and \(y\) is \(|x - y|\). Thus, the distance from \(-2\) to \(-4\) is \(|-2 - (-4)| = |-2 + 4| = |2| = 2\), and the distance from \(-2\) to 5 is \(|-2 - 5| = |-7| = 7\).

The following properties of the absolute value function need to be memorized.

**Lemma 1.** For any two real numbers \(x\) and \(y\), we have

\[
|xy| = |x| \ |y|.
\]

This equality can be verified by considering cases. One of the four possible cases is checked as follows: Suppose \(x < 0\) and \(y \geq 0\). Then \(xy\) is \(\leq 0\) and we have

\[
|xy| = - (xy) = (-x) y = |x| \ |y|.
\]

The other three cases are similarly checked.