Method of Undetermined Coefficients

by Andrew Binder

February 23, 2012

Problem. Solve the initial value problem \( y'' + 6y' + 13y = 9\cos(2t) - 87\sin(2t), \) \( y(0) = 9, \ y'(0) = -4. \)

Solution. In order to solve this second order differential equation, we will use the method of undetermined coefficients. The first step of the method of undetermined coefficients is to find the homogeneous solution to the differential equation. We will call the homogeneous solution \( y_h. \)

The homogeneous solution satisfies
\[ y''_h + 6y'_h + 13y_h = 0. \] (1)

Notice that the above equation is simply the left hand side of our initial problem set equal to zero. We can solve this homogeneous equation using either the factoring operator method or the characteristic equation method. I will use the characteristic equation method. The characteristic equation for this second order differential equation is
\[ r^2 + 6r + 13 = 0. \] (2)

The roots of the characteristic equation may be found using the quadratic equation. The roots are undetermined constants. In order to determine \( A \) and \( B, \) we will substitute \( y_p \) into Eq. (4) and solve for them. Before I do so, let me calculate the derivatives of \( y_p \) that appear in Eq. (4):
\[ y'_p = 2A \cos(2t) - 2B \sin(2t) \]
\[ y''_p = -4A \sin(2t) - 4B \cos(2t). \]

Now, substitute \( y_p \) and its derivatives into Eq. (4):
\[ [-4A \sin(2t) - 4B \cos(2t)] + 6[2A \cos(2t) - 2B \sin(2t)] + 13[A \sin(2t) + B \cos(2t)] = 9\cos(2t) - 87\sin(2t). \] (6)

Simplify:
\[ 9B + 12A = 9 \quad 9A - 12B = -87. \] (7)

From here, we set \( 9B + 12A = 9 \) and \( 9A - 12B = -87, \) and solve the system of linear equations. To get to that point, I set the coefficient of the sine function on the left hand side equal to the coefficient of the sine function on the right hand side. A similar procedure was used for the cosine terms. Solving this system of linear equations, you will find that \( A = -3 \) and \( B = 5. \) Therefore, our particular solution is
\[ y_p = -3\sin(2t) + 5\cos(2t). \] (8)