THEORY

DERIVATIVES AND DIFFERENTIATION

Definition: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

DERIVATIVE RULES

1. Sum and Difference: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
2. Scalar Multiple: $\frac{d}{dx}(cf(x)) = c \cdot f'(x)$
3. Product: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
   (Mnemonic: If $f$ is “hi” and $g$ is “lo”, then the product rule is “hi de hi plus hi de lo.”)
4. Quotient: $\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
   (Mnemonic: “Hi de ho minus ho de hi over ho de ho.”)
5. The Chain Rule
   - First formulation: if $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$
   - Second formulation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
6. Implicit differentiation: Used for curves when it is difficult to express $y$ as a function of $x$. Differentiate both sides of the equation with respect to $x$. Use the chain rule carefully whenever $y$ appears. Then, rewrite $\frac{dx}{dy}$ and solve for $y$.

COMMON DERIVATIVES

1. Constants: $\frac{d}{dx}(c) = 0$
2. Linear: $\frac{d}{dx}(ax + b) = a$
3. Powers: $\frac{d}{dx}(x^n) = nx^{n-1}$ (true for all real $n \neq 0$)
4. Polynomials: $\frac{d}{dx}((a_0x^n + a_1x^{n-1} + \ldots + a_nx + a_0)) = n a_0x^{n-1} + \ldots + 2a_2x + a_1$
5. Exponential
   - Base $e$: $\frac{d}{dx}(e^x) = e^x$
   - Arbitrary base: $\frac{d}{dx}(a^x) = a^x \ln a$
6. Logarithmic
   - Base $e$: $\frac{d}{dx}(\ln x) = \frac{1}{x}$
   - Arbitrary base: $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
7. Trigonometric
   - Sine: $\frac{d}{dx}(\sin x) = \cos x$
   - Cosine: $\frac{d}{dx}(\cos x) = -\sin x$
   - Tangent: $\frac{d}{dx}(\tan x) = \sec^2 x$
   - Cotangent: $\frac{d}{dx}(\cot x) = -\csc^2 x$
   - Secant: $\frac{d}{dx}(\sec x) = \sec x \tan x$
   - Cosecant: $\frac{d}{dx}(\csc x) = -\csc x \cot x$
8. Inverse Trigonometric
   - Arccosine: $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
   - Arctangent: $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
   - Arccotangent: $\frac{d}{dx}(\arccot x) = -\frac{1}{1+x^2}$
   - Arcsecant: $\frac{d}{dx}(\arcsec x) = \frac{1}{|x| \sqrt{x^2 - 1}}$
   - Arccosecant: $\frac{d}{dx}(\arccsc x) = -\frac{1}{|x| \sqrt{x^2 - 1}}$

INTEGRALS AND INTEGRATION

DEFINITE INTEGRAL

The definite integral $\int_a^b f(x) \, dx$ is the signed area between the function $y = f(x)$ and the $x$-axis from $x = a$ to $x = b$.

- Formal definition: Let $n$ be an integer and $\Delta x = \frac{b-a}{n}$. For each $k = 0, 2, \ldots, n-1$, pick point $x_k^* \in$ the interval $[a + k \Delta x, a + (k+1) \Delta x]$. The expression $\sum_{k=0}^{n} f(x_k^*) \Delta x$ is a Riemann sum. The definite integral $\int_a^b f(x) \, dx$ is defined as $\lim_{n \to \infty} \sum_{k=0}^{n} f(x_k^*) \Delta x$.

INDEFINITE INTEGRAL

- Antiderivative: The function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
- Indefinite integral: The indefinite integral $\int f(x) \, dx$ represents a family of antiderivatives: $F(x) + C$ if $F'(x) = f(x)$.

FUNDAMENTAL THEOREM OF CALCULUS

Part 1: If $f(x)$ is continuous on the interval $[a, b]$, then the area function $F(x) = \int_a^x f(t) \, dt$ is continuous and differentiable on the interval and $F'(x) = f(x)$.

Part 2: If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x) \, dx = F(b) - F(a)$.

APPROXIMATING DEFINITE INTEGRALS

1. Left-hand rectangle approximation: $L_n = f(x_0) + f(x_1) + \ldots + f(x_{n-1})$
2. Right-hand rectangle approximation: $R_n = f(x_1) + f(x_2) + \ldots + f(x_n)$
3. Midpoint Rule: $M_n = \frac{f(x_0) + f(x_1) + \ldots + f(x_n)}{n}$
4. Trapezoidal Rule: $T_n = \frac{f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)}{n}$
5. Simpson’s Rule: $S_n = \frac{f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{3n}$

TECHNIQUES OF INTEGRATION

1. Properties of Integrals
   - Sums and differences: $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$
   - Constant multiples: $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$
   - Definite integral comparison: $\int_a^b 2f(x) \, dx = 2 \int_a^b f(x) \, dx$

2. Integration by Parts
   - Best used to integrate a product when one factor ($u = f(x)$) has a simple derivative and the other factor ($dv = g(x) \, dx$) is easy to integrate.
   - Indefinite integrals: $\int u dv = uv - \int v du$ where $uv$ is easy to integrate
   - Definite integrals: $\int_a^b u dv = [uv]_a^b - \int_a^b v du$

3. Trigonometric Substitutions: Used to integrate expressions of the form $\sqrt{a^2 + x^2}$

APPLICATIONS

GEOMETRY

Area: $\int_a^b (f(x) - g(x)) \, dx$ is the area bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$

Volume of revolved solid (disk method): $\pi \int_a^b [f(x)]^2 \, dx$ is the volume of the solid swept out by the curve $y = f(x)$ as it revolves around the $x$-axis on the interval $[a, b]$.

Volume of revolved solid (washer method): $\pi \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$ is the volume of the solid swept out between $y = f(x)$ and $y = g(x)$ as they revolve around the $x$-axis on the interval $[a, b]$ if $f(x) \geq g(x)$.

Surface area: $2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx$ is the area of the surface swept out by revolving the function $y = f(x)$ about the $x$-axis between $x = a$ and $x = b$.

Volume of revolved solid (shell method): $2\pi \int_a^b x f(x) \, dx$ is the volume of the solid obtained by revolving the region under the curve $y = f(x)$ between $x = a$ and $x = b$ around the $y$-axis.

Arc length: $\int_a^b \sqrt{1 + (f'(x))^2} \, dx$ is the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

Area of the surface from $x = a$ to $x = b$.