We have also learnt in Chapter 1 that if $f: X \to Y$ such that $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \text{ o } f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \text{ o } f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse

Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \dfrac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

Example 1 Find the principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \dfrac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ is $\dfrac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\dfrac{2\pi}{3}\right) = \dfrac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ is $\dfrac{2\pi}{3}$

EXERCISE 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\dfrac{1}{2}\right)$ **2.** $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ **3.** $\operatorname{cosec}^{-1}(2)$

4. $\tan^{-1}(-\sqrt{3})$ **5.** $\cos^{-1}\left(-\dfrac{1}{2}\right)$ **6.** $\tan^{-1}(-1)$

Also $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \cos^{-1}\dfrac{1-\tan^2 y}{1+\tan^2 y} = \cos^{-1}(\cos 2y) = 2y = 2\tan^{-1} x$

(iii) Can be worked out similarly.

We now consider some examples.

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x, \; -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x, \; \dfrac{1}{\sqrt{2}} \le x \le 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1} x = \theta$. We have

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= 2\sin^{-1} x$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$

Example 4 Show that $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{2}{11} = \tan^{-1}\dfrac{3}{4}$

Solution By property 5 (i), we have

$$\text{L.H.S.} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$$

Example 5 Express $\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right), \; -\dfrac{3\pi}{2} < x < \dfrac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$