1.3 Multiplying Whole Numbers

Multiplication is really just repeated addition. When we say "4 times 3 equals 12," we can think of it as starting at 0 and adding 3 four times over:

$$0 + 3 + 3 + 3 + 3 = 12.$$

We can leave out the 0, since 0 is the additive identity ($0+3 = 3$). Using the symbol $\times$ for multiplication, we write

$$3 + 3 + 3 + 3 = 4 \times 3 = 12.$$

The result of multiplying two or more numbers is called the **product** of the numbers. Instead of the $\times$ symbol, we often use a **central dot** ($\cdot$) to indicate a product. Thus, for example, instead of $2 \times 4 = 8$, we can write

$$2 \cdot 4 = 8.$$

Example 12. We assume you remember the products of small whole numbers, so it should be easy for you to reproduce the partial *multiplication table* below. For example, the number in the row labelled **7** and the column labelled **5** is the product $7 \cdot 5 = 35$.

×	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

Table 1.3: Multiplication table

1. Do you notice any patterns or regularities in the multiplication table? Can you explain them?
2. Why does the second row from the top contain only 0's?
3. Why is the third row from the top identical to the first row?
4. Is multiplication commutative? How can you tell from the table?

and then performing the addition

$$\begin{array}{rr} & 2\,4 \\ \times & 3\,2 \\ \hline & 4\,8 \\ + & 7\,2\,0 \\ \hline & 7\,6\,8 \end{array}$$

Here's another example.

Example 18. Find the product of 29 and 135.

Solution. We choose 29 as the multiplier since it has the fewest digits.

$$\begin{array}{rr} & 1\,3\,5 \\ \times & 2\,9 \\ \hline \end{array}$$

We use the 1-digit multiplier 9 to obtain the first partial product

$$\begin{array}{rr} & \{3\}\{4\} \\ & 1\;\;3\;\;5 \\ \times & 2\;\;9 \\ \hline & 1\,2\;\;1\;\;5 \end{array}$$

Notice that we put down 5 and carried 4 to the *tens* place, and also put down 1 and carried 3 to the *hundreds* place. Next we use the 1-digit multiplier 2 (standing for 2 *tens*) to obtain the second partial product, shifted left by putting a 0 in the *ones* place

$$\begin{array}{rr} & \{1\} \\ & 1\,3\,5 \\ \times & 2\,9 \\ \hline & 1\,2\,1\,5 \\ & 2\,7\,0\,0 \end{array}$$

(What carry did we perform?) Finally, we add the partial products to obtain the (total) product

$$\begin{array}{rr} & 1\,3\,5 \\ \times & 2\,9 \\ \hline & 1\,2\,1\,5 \\ + & 2\,7\,0\,0 \\ \hline & 3\,9\,1\,5 \end{array}$$

Note that the whole procedure is compactly recorded in the last step, which is all that you need to write down. □

1.6 Order of operations

We often do calculations that involve more than one operation. For example

$$1 + 2 \times 3$$

involves both addition and multiplication. Which do we do first? If we do the multiplication first, the result is $1 + 6 = 7$, and if we do the addition first, the result is $3 \times 3 = 9$. Obviously, if we want the expression

$$1 + 2 \times 3$$

to have a definite and unambiguous meaning, we need a convention or agreement about the order of operations. It could have been otherwise, but the convention in this case is:

multiplication before addition.

With this convention, when I write $1 + 2 \times 3$, you know that I mean 7 (not 9). The precedence of multiplication can be made explicit using the **grouping symbols** () (parentheses):

$$1 + (2 \times 3) = 1 + 6 = 7.$$

If one of us insists that the addition be done first, we can do that by re-setting the parentheses:

$$(1 + 2) \times 3 = 3 \times 3 = 9.$$

Thus grouping symbols can be used to force any desired order of operations. Common grouping symbols, besides parentheses, are brackets, [ ], and braces, { }. The square root symbol $\sqrt{\ }$ is also a grouping symbol. For example

$$\sqrt{4 + 5} = \sqrt{9} = 3.$$

The $\sqrt{\ }$ symbol acts like a pair of parentheses, telling us to evaluate what is inside (in this case, the sum $4 + 5$) first, *before* taking the square root.

The **order of operations** is:

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

"In order of appearance" means in order from left to right. Thus in the expression

$$2 + 5 - 3,$$

the addition comes first, so it is evaluated first,

$$2 + 5 - 3 = 7 - 3 = 4,$$

while in the expression

$$8 - 6 + 11,$$

the subtraction is done first because it comes first,

$$8 - 6 + 11 = 2 + 11 = 13.$$

The rectangle has length 6 cm and width 2 cm, so it has area $6 \cdot 2 = 12$ cm^2. Both triangles have legs of length 1 and 2, so the area of each is $(2 \cdot 1) \div 2 = 1$ cm^2. Adding the three areas gives the total area of the polygon: $1 + 12 + 1 = 14$ cm^2. To find the perimeter, we use the Pythagorean theorem to find the lengths of the two slanted sides:

$$c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

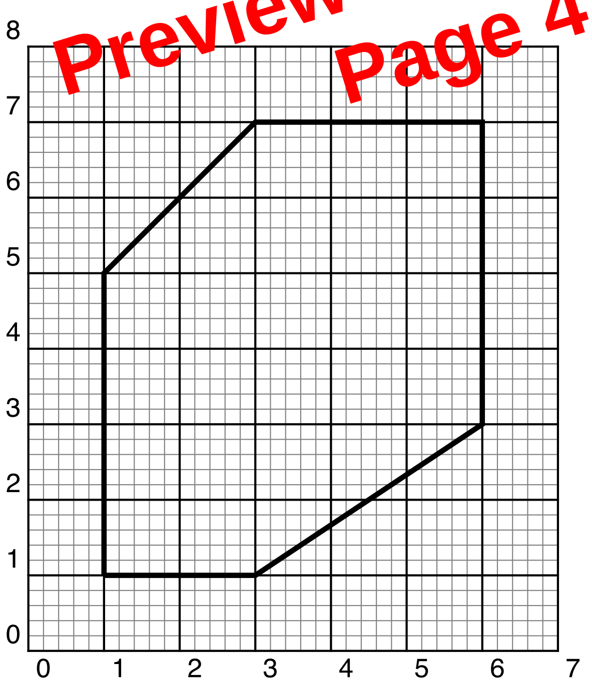
Thus the perimeter is

$$\sqrt{5} + 7 + \sqrt{5} + 7 = 14 + 2\sqrt{5} \text{ cm}.$$

Since $\sqrt{5}$ is not an integer, we do not simplify the expression further. But we can make the following estimate: Since $\sqrt{5}$ is between 2 and 3, $2\sqrt{5}$ is between 4 and 6, and it follows that the perimeter is between 18 and 20 centimeters. □

1.8.1 Exercises

1. Find the area of a right triangle whose legs are 5 in and 12 in.
2. Find the perimeter of the right triangle in the previous example.
3. Find the area of a rectangle whose length is 8 ft and whose width is 7 feet.
4. Find the perimeter of the rectangle in the previous example.
5. Find the length of the diagonal of a square which is 2 cm on a side. Between what two whole numbers does the answer lie?
6. Find the area and perimeter of the polygon below.



Solution.

$$8\frac{2}{3} = \frac{8 \cdot 3 + 2}{3} = \frac{24 + 2}{3} = \frac{26}{3}.$$

□

Example 50. Write the whole number 8 as an improper fraction in three different ways.

Solution. We can think of the whole number 8 as the "mixed" number $8\frac{0}{q}$ for any nonzero q. By the boxed rule,

$$8 = \frac{8 \cdot q + 0}{q} = \frac{8 \cdot q}{q}.$$

Picking three nonzero numbers for q, say, $2, 5$ and 10, we get

$$8 = \frac{8 \cdot 2}{2} = \frac{16}{2}, \qquad 8 = \frac{8 \cdot 5}{5} = \frac{40}{5}, \qquad \text{and} \quad 8 = \frac{8 \cdot 10}{10} = \frac{80}{10}.$$

□

2.3.4 Exercises

Convert the mixed numbers into improper fractions:

1. $1\frac{1}{2}$
2. $8\frac{1}{3}$
3. $15\frac{3}{8}$
4. $5\frac{3}{4}$
5. $11\frac{5}{6}$

Using $N = \frac{N \cdot q}{q}$ for any nonzero q, write three fractions equal to each given whole number:

6. 5
7. 11
8. 10

Example 62. (a) Find the GCF of the set $\{60, 135, 150\}$. (b) Find the GCF of the subset $\{60, 150\}$.

Solution. (a) Following the boxed procedure:

1. the prime factorizations are

$$\begin{aligned} 60 &= 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5 \\ 135 &= 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5 \\ 150 &= 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2 \end{aligned}$$

2. the common prime factors are 3 and 5;
3. (does not apply to this example);
4. the smallest exponent on 3 is 1 (in the factorizations of 60 and 150); the smallest exponent on 5 is also 1 (in the factorizations of 60 and 135);
5. the GCF is $3^1 \cdot 5^1 = 15$.

(b) For the two-number subset $\{60, 150\}$, the common prime factors are 2, 3 and 5. The smallest exponent on all three factors is 1. So the GCF is $2^1 \cdot 3^1 \cdot 5^1 = 30$.

Can you explain why the GCF in part (b) is bigger than in part (a)? □

2.6.3 Exercises

Find the GCF of each of the following sets of numbers:

1. $\{72, 48\}$
2. $\{72, 48, 36\}$
3. $\{72, 36\}$
4. $\{48, 36\}$
5. $\{36, 15\}$
6. $\{36, 14\}$
7. $\{15, 14\}$

2.6.4 Cancelling the GCF for lowest terms

Knowing that the GCF of $\{60, 150\} = 30$ allows us to reduce the fraction $\frac{60}{150}$ to lowest terms in one step: we simply cancel it out. Thus,

$$\frac{60}{150} = \frac{\cancel{60}^{\,2}}{\cancel{150}^{\,5}} = \frac{2}{5} \qquad \text{(cancelling the GCF, 30).}$$

Recall that this is short-hand for

$$\frac{60 \div 30}{150 \div 30} = \frac{2}{5}.$$

2.7.1 Exercises

Find the products, using pre-cancellation where possible. Check that the answers are in lowest terms. Express any improper fractions as mixed numbers.

1. $\frac{4}{5} \cdot \frac{7}{12}$
2. $\frac{32}{45} \cdot \frac{9}{16} \cdot \frac{15}{6}$
3. $12 \cdot \frac{5}{8} \cdot \frac{2}{3}$
4. $\frac{9}{22} \cdot \frac{21}{12} \cdot 2\frac{4}{7}$
5. $\frac{2}{3}$ of 24.
6. $\frac{3}{4}$ of $\frac{2}{3}$ of 50.
7. $2\frac{2}{3} \cdot 1\frac{3}{4}$.

Use multiplication to answer the following questions.

8. What is the area of a rectangle with length $2\frac{1}{2}$ feet and width $2\frac{3}{10}$ feet?
9. A $12\frac{1}{2}$-gallon fish tank is only three-fifths full. How many gallons of water must be added to fill it up?
10. $2\frac{2}{3}$ cups of beans are needed to make 4 bowls of chili. How many cups are needed to make 8 bowls? 1 bowl? 3 bowls?

2.8 Adding and Subtracting Fractions

If I eat a third of a pizza for lunch, and another third for dinner, then I have eaten two thirds in total. That is,

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Similar logic applies whenever we add two (or more) fractions *with the same denominator* – we simply add the numerators, while keeping the denominator fixed:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

A very similar rule holds for subtraction of fractions with the same denominator:

The LCM can also be found using prime factorizations. This is useful when the numbers are rather large.

Example 73. Find the LCM of the two-number set $\{a, b\}$, where a and b have the following prime factorizations.

$$a = 2^4 \cdot 3^2 \cdot 7 \quad \text{and} \quad b = 2 \cdot 3^4 \cdot 11.$$

Solution. The LCM must be a multiple of both numbers, so that it must be divisible by the highest power of every prime factor that appears in any one of the factorizations. The prime factors of a and b are 2, 3, 7 and 11. The highest powers that appear are

$$2^4, \quad 3^4, \quad 7^1, \quad 11^1.$$

The LCM is the product of these powers:

$$\text{LCM}\{a, b\} = 2^4 \cdot 3^4 \cdot 7 \cdot 11.$$

This is a rather large number, but it is the smallest which is a multiple of both a and b. Notice that the actual values of a and b, and their LCM (which we could calculate by multiplication) were not needed – only their prime factorizations. □

Here is a summary of the procedure:

To find the LCM of a set of numbers:

1. find the prime factorization of each number;
2. for each prime factor, find the largest exponent that appears on it in any of the factorizations;
3. the LCM is the product of the prime factors with the exponents found in step 2.

Compare this procedure with the procedure for finding the GCF of a set of numbers. There are similarities and significant differences.

2.8.4 Exercises

Find the LCM of the following sets of numbers. Use the prime factorization method for larger numbers.

1. $\text{LCM}\{25, 10\}$
2. $\text{LCM}\{48, 60\}$
3. $\text{LCM}\{10, 15, 25\}$
4. $\text{LCM}\{8, 12\}$
5. $\text{LCM}\{9, 6, 12\}$
6. $\text{LCM}\{60, 168\}$

7. $\text{LCM}\{51, 34, 17\}$
8. $\text{LCM}\{15, 12\}$
9. $\text{LCM}\{18, 8\}$
10. $\text{LCM}\{3, 4, 5\}$
11. $\text{LCM}\{4, 14\}$
12. $\text{LCM}\{2, 5, 9\}$
13. $\text{LCM}\{3^2 \cdot 5^2 \cdot 11, \quad 3^4 \cdot 7^2\}$

2.8.5 The LCD

To add unlike fractions so that the sum is in lowest terms, we use the LCM of their denominators. This is such a frequent operation with fractions that it has its own name: the LCD or **Least Common Denominator**.

Example 74. Find the LCD of the fractions

$$\frac{1}{8}, \quad \frac{3}{10}, \quad \text{and} \quad \frac{1}{18}.$$

Solution. The LCD of the fractions is the LCM of their denominators

$$\text{LCM}\{8, 10, 18\}.$$

Looking at the prime factorizations

$$8 = 2^3, \qquad 10 = 2 \cdot 5, \qquad 18 = 2 \cdot 3^2,$$

and taking the highest power of each prime that occurs, we see that the LCM is

$$2^3 \cdot 3^2 \cdot 5 = 360.$$

This is LCD of the fractions. □

Example 75. Find the sum of the unlike fractions $\frac{1}{8} + \frac{3}{10} + \frac{1}{18}$.

Solution. The LCD is the LCM from the previous example: 360. Now we observe that

$$360 \quad = \quad 8 \cdot 45 \quad = \quad 10 \cdot 36 \quad = \quad 18 \cdot 20.$$

Thus

$$\frac{1}{8} = \frac{1 \cdot 45}{8 \cdot 45}, \qquad \frac{3}{10} = \frac{3 \cdot 36}{10 \cdot 36} \qquad \text{and} \qquad \frac{1}{18} = \frac{1 \cdot 20}{18 \cdot 20}.$$

It follows that

$$\begin{aligned} \frac{1}{8} + \frac{3}{10} + \frac{1}{18} &= \frac{1 \cdot 45}{8 \cdot 45} + \frac{3 \cdot 36}{10 \cdot 36} + \frac{1 \cdot 20}{18 \cdot 20} \\ &= \frac{45 + 108 + 20}{360} \\ &= \frac{173}{360}. \end{aligned} \tag{2.2}$$

Since the 173 is not divisible by 2, 3, or 5, the fraction is in lowest terms. □

and it follows that the corresponding order of the mixed numbers (each with whole number part 1) is

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5}.$$

Adjoining the smallest (proper) fraction $\frac{11}{12}$ at the end, we have, finally, the decreasing order

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5} > \frac{11}{12}.$$

□

2.9.1 Exercises

1. Arrange in decreasing order: $\frac{4}{9}$, $\frac{3}{8}$, and $\frac{1}{3}$.
2. Arrange in increasing order: $\frac{3}{5}$, $\frac{3}{4}$, and $\frac{5}{7}$.
3. Is $\frac{5}{12}$ of an inch more or less than $\frac{7}{16}$ of an inch?
4. Is $9\frac{5}{8}$ inches closer to 9 or 10 inches?
5. A stock price changed from $5\frac{5}{8}$ to $5\frac{3}{4}$. Did the price go up or down?

2.10 Division of Fractions

Our intuition is not very good when we think of dividing two fractions. How many times does $\frac{2}{3}$ "go into" $\frac{3}{4}$, for example? Equivalently, what fraction results from the division problem

$$\frac{3}{4} \div \frac{2}{3}?$$

It is not at all obvious. We will derive a simple rule, but we'll need an auxiliary concept.

2.10.1 Reciprocals

Two non-zero numbers are **reciprocal** if their product is 1. Thus, if x and y are nonzero numbers, and if

$$x \cdot y = 1,$$

then x *is the reciprocal of* y, and, also, y *is the reciprocal of* x.

The rule for multiplying fractions, together with obvious cancellations, shows that

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{\cancel{a}^{1} \cdot b}{b \cdot \cancel{a}^{1}} = \frac{1 \cdot \cancel{b}^{1}}{\cancel{b}^{1} \cdot 1} = \frac{1}{1} = 1.$$

This means that

Chapter 3

Decimals and Percents

Decimals are mixed numbers in which the proper fractional part has a denominator which is 10, or 100, or 1000, or, more generally, a *power* of 10. Here are some powers of 10:

$$
\begin{aligned}
10^0 &= 1 \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1000 \\
10^4 &= 10000 \\
10^n &= 1 \text{ followed by } n \text{ 0's}
\end{aligned}
$$

The number above and to the right of 10 is the *exponent*, or *power*, and indicates the number of times that 10 appears as a factor. Thus

$$10^3 = 10 \times 10 \times 10 = 1000,$$

and so forth.

Here are some examples of mixed numbers in which the denominator of the fractional part is a power of ten, together with their representations as decimals:

$$
\begin{aligned}
1\frac{3}{10} &= 1.3 \\
21\frac{37}{100} &= 21.37 \\
13\frac{21}{1000} &= 13.021
\end{aligned}
$$

The *decimal point* (which looks like the period at the end of a sentence) separates the whole number part from the proper fractional part. The digits to the right of the decimal point represent the fractional part according to the following rule:

- the digits to the right of the decimal point constitute the numerator;
- the *number* of digits to the right of the decimal point is the power of 10 which constitutes the denominator.

There is no need to show the fraction bar. As we shall see, this makes computation very easy. There are some subtleties to watch out for, however. In the example

$$13\frac{21}{1000} \quad = \quad 13.021,$$

we had to use 021 to represent the numerator 21 because we need *three* digits to show that the denominator is 10^3.

3.1 Decimal place values

In Chapter 1 we showed that any whole number can be written using just the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, by using the *place value* system. Thus, you will recall that

4267 stands for 4 *thousands* + 2 *hundreds* + 6 *tens* + 7 *ones*.

Using powers of 10, we can write this more compactly:

$$4267 = 4 \times 10^3 \ + \ 2 \times 10^2 \ + \ 6 \times 10^1 \ + \ 7 \times 10^0.$$

The decimal point allows us to adjoin more places (to the right of the decimal point), with place values less than 1. Thus

23.59 stands for 2 *tens* + 3 *ones* + 5 *tenths* + 9 *hundredths*.

The first place to the right of the decimal point has the place value $\frac{1}{10}$, or one *tenth*; the second place to the right of the decimal point has the place value $\frac{1}{100}$, or one *hundredth*, and so forth. More generally,

> The nth place to the right of the decimal point has the place value $\dfrac{1}{10^n}$.

This is a continuation of the pattern we established in Chapter 1: as we move to the right, place values decrease ten-fold (by a factor of *one tenth*). If we adopt the convention that

$$10^{-n} = \frac{1}{10^n},$$

then the sequence of place values, from left to right, is just the sequence of decreasing powers of ten:

$$\ldots, 10^3, \quad 10^2, \quad 10^1, \quad 10^0 \quad \text{(decimal point)} \quad 10^{-1}, \quad 10^{-2}, \quad 10^{-3}, \quad \ldots$$

…, *thousands, hundreds, tens, ones* (decimal point) *tenths, hundredths, thousandths,* …

Example 97. Name the digit in tens place, and the digit in the hundredths place, in the decimal 304.671.

Solution. The tens (10^1) place is the second place to the left of the decimal point. The digit in that place is 0. The hundredths (10^{-2}) place is the second place to the right of the decimal point. The digit in that place is 7. □

Example 98. What is the digit in the ten-thousandths place of the decimal 0.00286?

Solution. One ten-thousandth is

$$\frac{1}{10000} = \frac{1}{10^4},$$

so the ten-thousandths place is the fourth place to the right of the decimal point. The digit in that place is 8. □

This implies that the number of decimal places in the product of two (or more) decimals is the sum of the numbers of decimal places in the factors. Thus any set of decimals can be multiplied by following a two step procedure:

> To multiply two or more decimals:
>
> - Multiply the decimals as if they were whole numbers, ignoring the decimal points (this gives the numerator of the decimal fraction);
> - Add the number of decimal places in each factor (this gives the denominator of the decimal fraction by specifying the number of decimal places).

Example 114. Find the product 21.02×0.004.

Solution. Temporarily ignoring the decimal points, we multiply $2102 \times 4 = 8408$. Now 21.02 has two decimal places, and 0.004 has three. So the product will have $2 + 3 = 5$ decimal places. In other words,

$$21.02 \times 0.004 = 8408 \div 10^5 = 0.08408.$$

□

Example 115. Find the product of 12, 0.3, and 0.004.

Solution. Ignoring the decimal points, the product is $12 \times 3 \times 4 = 144$. The numbers of decimal places in the three decimals are, left to right, 0, 1 and 3, which add up to 4. Thus,

$$12 \times 0.3 \times 0.004 = 144 \div 10^4 = 0.0144.$$

□

3.7.1 Exercises

Find the following products.

1. 68.4×23
2. 804×6.2
3. 26.09×0.004
4. 4.09×93
5. 100×9.9
6. 14.093×6.39
7. 64.9×0.345
8. 0.0001×0.001
9. 1000×0.053
10. $6 \times 0.9 \times 0.02 \times 0.001$

```
          .2909
     ___________
 11 |  3.20
      -2 2
     -----------
        1 00
      -   99
     -----------
          100
       -   99
     -----------
             1
```

It is evident that these last two steps will now repeat again and again, forever. The decimal will never terminate, since a 0 remainder will never occur. Still, it is easy to describe the quotient: it will consist of 29 after the decimal point, followed by an endless string of 09's. To indicate this, we put a bar over the repeated string:

$$0.2909090909090909\cdots = 0.29\overline{09}.$$

Thus, $3.2 \div 11 = 0.29\overline{09}$. □

We emphasize that it may take a while for the digits to start repeating, and the bar is only placed over the repeating part. For example, the repeating decimal

$$0.796812341234123412341234\cdots$$

is written, using the bar notation, as

$$0.7968\overline{1234}.$$

The repeated string can be quite long, or just a single digit. Here are some examples, which you should verify by carrying out the division process:

$$3 \div 7 = 0.\overline{428571}$$
$$2 \div 11 = 0.\overline{09}$$
$$5 \div 3 = 1.\overline{6}$$

Sometimes, we don't care whether a decimal terminates or repeats. For example, if we know in advance that we can round off our answer to a given decimal place, we carry out the division only as far as the right neighboring place and stop (this gives us all we need to know to round off).

Example 118. Suppose you bought 900 buttons for \$421. To the nearest cent, how much did you spend for each button?

Solution. We need to perform the division $421 \div 900$. Since we are rounding to the nearest *hundredth* (cent), we need only carry out the division as far as the *thousandths* place. Since the thousandths place is the the third place to the right of the decimal point, we adjoin three insignificant 0's to the dividend, setting up the division as $421.000 \div 900$. We omit the details, but you can verify that the quotient, carried out to three decimal places, is 0.467, which, rounded to the nearest hundredth, is 0.47. Therefore, the buttons cost approximately 47 cents each.

If you carried out the division process a little further, you noticed that the quotient is actually a repeating decimal:

$$421 \div 900 = 0.46777777\cdots = 0.46\overline{7}.$$

□

3.10 Percents, Conversions

A *percent* is a fraction in which the denominator is 100. The word "percent" comes from the Latin phrase *per centum* meaning "out of 100," and is symbolized by %. For example,

$$97\% \quad \text{means} \quad \frac{97}{100} \quad \text{or} \quad 0.97.$$

Percents need not be whole numbers, and they need not represent proper fractions. For example,

$$150\% = \frac{150}{100} = 1.5$$
$$0.5\% = \frac{0.5}{100} = 0.005$$

Fractions, decimals, and percents represent the same quantities in different ways, and we need to know how to convert one to another.

A decimal can be converted to a percent by moving the decimal point two places to the right and adjoining the % symbol. This is the same as multiplying the decimal fraction by 100, which exactly cancels the denominator, leaving just the numerator (the percent). Thus

$$0.68 = 68\%$$
$$2.05 = 205\%$$
$$0.708 = 70.8\%$$
$$1.4 = 140\%$$
$$0.0067 = 0.67\%$$

To convert a percent back into a decimal, we simply divide by 100, which, as we know, is equivalent to moving the decimal point two places to the left. Thus

$$92\% = 0.92$$
$$0.2\% = 0.002$$
$$138\% = 1.38$$
$$71.02\% = 0.7102$$

To convert a percent to a fraction (or mixed number), first convert the percent to a decimal, as above, then express the decimal as a fraction (with a visible denominator), and finally, reduce the fraction to lowest terms, if needed.

Example 120. Convert 10.8% into a fraction in lowest terms.

Solution. We first write 10.8% as a decimal.

$$10.8\% = 0.108.$$

Then, we write the decimal as a fraction with a visible denominator. In this case, because there are 3 decimal places, the denominator is 1000.

$$0.108 = \frac{108}{1000}.$$

6. A \$500 television is being sold at a 15% discount. What is the sale price?
7. Angela gets a 5% raise. Her original salary was $\$36,000$ per year. What is her new salary?
8. A car loses $\frac{2}{5}$ of its value over a period of years. If the car originally sold for $\$12,500$, what it would it sell for now?
9. On a test, Jose answers $\frac{7}{8}$ of the problems correctly. If there were 24 problems on the test, how many did he get wrong?
10. Medical expenses can be deducted from a person's income tax if they exceed 2% of total income. If Maribel's medical expenses were \$550, and her total income was $\$28,000$, can she deduct her medical expenses?

Chapter 4

Ratio and Proportion

In this chapter we develop another interpretation of fractions, as comparisons between two quantities.

4.1 Ratio

If a team wins ten games and loses five, we say that the **ratio** of wins to losses is 2 : 1 or "2 to 1." Where did the numbers 2 : 1 come from? We simply made a fraction whose numerator is the number of games the team won, and whose denominator is the number of games they lost, and reduced it to lowest terms: $\frac{\text{wins}}{\text{losses}} = \frac{10}{5} = \frac{2}{1}$.

We can do the same for any two quantities, a and b, as long as $b \neq 0$.

> The ratio of a to b ($b \neq 0$) is the fraction $\frac{a}{b}$, reduced to lowest terms.

Example 126. Find (a) the ratio of 9 to 18, (b) the ratio of 21 to 12, (c) the ratio of 64 to 4.

Solution. (a) The ratio of 9 to 18 is

$$\frac{9}{18} = \frac{1}{2} \quad \text{or} \quad 1 : 2.$$

(b) The ratio of 21 to 12 is

$$\frac{21}{12} = \frac{7}{4} \quad \text{or} \quad 7 : 4.$$

(c) The ratio of 64 to 4 is

$$\frac{64}{4} = \frac{16}{1} \quad \text{or} \quad 16 : 1.$$

□

Notice that in (c) we left the denominator 1 (rather than just writing 16) to maintain the idea of a *comparison* between two numbers.

We can form ratios of non-whole numbers. When the ratio is expressed in lowest terms, however, it is always the ratio of two whole numbers, as small as possible. This is the main point of a ratio comparison: if the given two numbers were *small whole numbers*, how would they compare?

Example 127. What is the ratio of $3\frac{3}{4}$ to $1\frac{1}{2}$?

Example 132. A printed page $8\frac{1}{2}$ inches in width has a margin $\frac{5}{8}$ inches wide on either side. Text is printed between the margins. What is the ratio of the width of the printed text to the total margin width?

Solution. The total margin width is $\frac{5}{8} + \frac{5}{8} = \frac{5}{4} = 1\frac{1}{4}$ inches (taking into account both left and right margins.) The width of the printed text is the difference

$$\text{total page width} - \text{total margin width} = 8\frac{1}{2} - 1\frac{1}{4} = 7\frac{1}{4} \text{ inches.}$$

The ratio of the width of the printed text to total margin width is

$$7\frac{1}{4} \div 1\frac{1}{4} = \frac{29}{4} \div \frac{5}{4} = \frac{29}{4} \cdot \frac{4}{5} = \frac{29}{5},$$

or 29 : 5. □

4.1.1 Exercises

Find the ratios.

1. 14 to 4
2. 30 to 32
3. 56 to 21
4. $1\frac{5}{8}$ to $3\frac{1}{4}$
5. $2\frac{1}{12}$ to $2\frac{1}{4}$
6. 14.4 to 5.4
7. 1.69 to 2.6
8. 3 hours to 40 minutes
9. 8 inches to $5\frac{1}{2}$ feet
10. In the late afternoon, a 35 foot tree casts an 84 foot shadow. What is the ratio of the tree's height to the shadow's length?

4.2 Proportions

A **proportion** is a statement that two ratios are equal. Thus

$$\frac{40}{20} = \frac{10}{5}$$

is a proportion, because both ratios are equivalent to the ratio 2 : 1 (= the fraction $\frac{2}{1}$).

4.2.3 Exercises

Solve the following proportions.

1. $\dfrac{1}{5} = \dfrac{3}{x}$
2. $\dfrac{15}{y} = \dfrac{2}{3}$
3. $\dfrac{100}{5} = \dfrac{20}{y}$
4. $\dfrac{A}{9} = \dfrac{5}{3}$
5. $\dfrac{11}{B} = \dfrac{1}{2}$
6. $\dfrac{5}{3} = \dfrac{c}{6}$
7. $\dfrac{s}{3} = \dfrac{4}{13}$
8. $\dfrac{1.2}{7} = \dfrac{A}{0.84}$
9. $\dfrac{P}{100} = \dfrac{75}{125}$
10. $\dfrac{3\frac{1}{5}}{x} = \dfrac{4}{2\frac{1}{2}}$

4.3 Percent problems

Any problem involving percent can be stated (or restated) in the form

"A is P percent of B"

where one of the numbers A, B or P is unknown. We can make this into a mathematical equation by making the following "translations:"

$$\text{"is"} \longleftrightarrow =$$
$$\text{"}P\text{ percent"} \longleftrightarrow \frac{P}{100}$$
$$\text{"of"} \longleftrightarrow \times$$

This gives us the equation

$$A = \frac{P}{100} \times B.$$

If we divide both sides of the equation by B, we obtain the proportion in the box below.

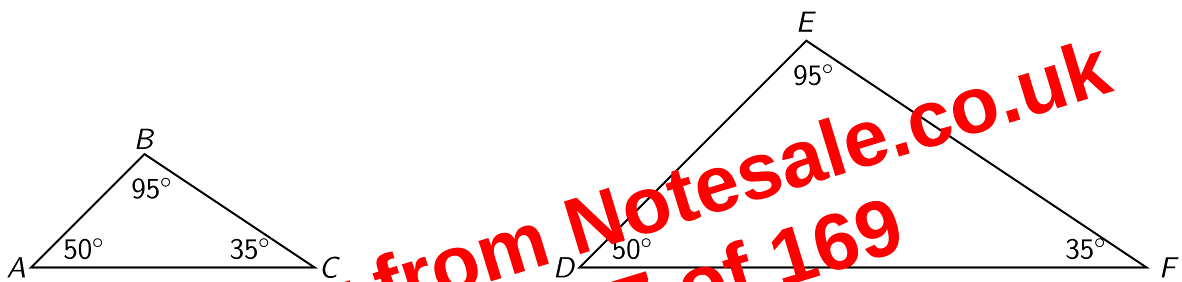
4. In a sample of 600 bottles, 11 were found to be leaking. Approximately how many bottles would you expect to be leaking in a sample of 20,000 bottles?

5. The ratio of the weight of lead to the weight of an equal volume of aluminum is 21 : 5. If a bar of aluminum weighs 15 pounds, how much would a bar of lead of the same size weigh?

4.5 Similar triangles

Two triangles which have the same shape but possibly different sizes are called **similar**. Having the same shape means that the three angles of one triangle are equal to the three corresponding angles in the other.

The triangles below are similar, because the angle at A in the small triangle is equal to the angle at D in the big triangle ($50°$); the angle at B in the small triangle is equal to the angle at E in the big triangle ($95°$); and similarly the angle at C in the small triangle is equal to the angle at F in the large triangle ($35°$).



Given any triangle, we can obtain a similar one by enlarging (or reducing) all the side lengths in a fixed ratio. In the example above, the bigger triangle was obtained from the smaller using the enlargement ratio 2 : 1. If $\overline{AB}$ stands for the length of the side AB, etc., then

$$\overline{DE} = 2 \times \overline{AB}, \qquad \overline{EF} = 2 \times \overline{BC}, \qquad \text{and} \quad \overline{DF} = 2 \times \overline{AC}.$$

Equivalently,

$$\frac{\overline{DE}}{\overline{AB}} = \frac{\overline{EF}}{\overline{BC}} = \frac{\overline{DF}}{\overline{AC}} = \frac{2}{1}.$$

In words: sides which are opposite equal angles (called **corresponding sides**) have length ratio equal to the ratio 2 : 1. We must be careful to compare side lengths in the proper order. In this case, we used the order larger : smaller, since that is the order in the ratio 2 : 1. We could have used the reverse order, smaller : larger, but only if we also used the reversed ratio 1 : 2.

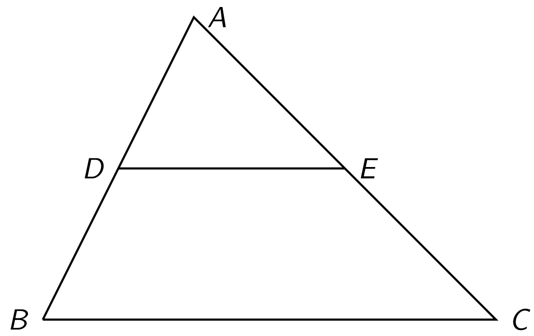
If we know, say, that $\overline{BC}$ is 12 feet, we can solve the proportion

$$\frac{2}{1} = \frac{\overline{EF}}{12}$$

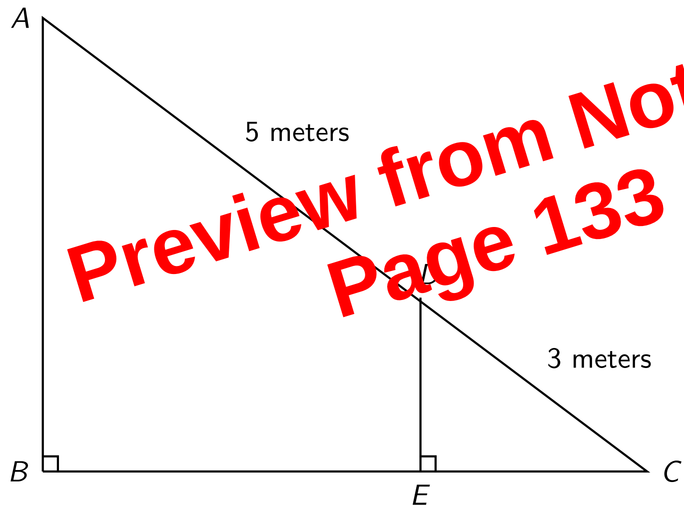
and determine that $\overline{EF}$ is 24 feet.

This is the key fact about similar triangles.

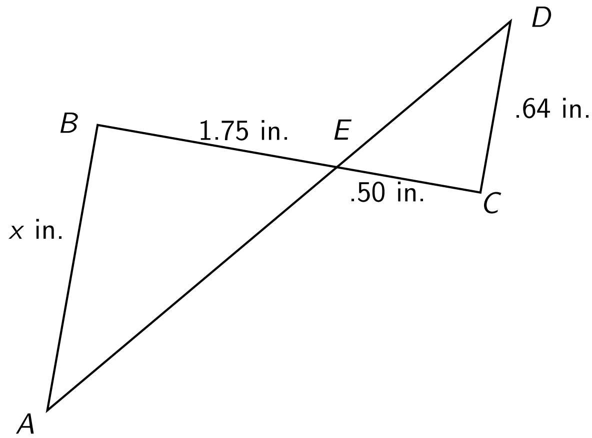
6. In the figure, $\overline{DE} \parallel \overline{BC}$. D cuts $\overline{AB}$ exactly in half. If $\overline{BC}$ is 12 centimeters long, how long is $\overline{DE}$?



7. In the figure below, $\overline{AB} \parallel \overline{DE}$, and length measures are given in meters. How long is $\overline{AB}$ if $\overline{DE}$ has a length of $2\frac{2}{5}$ meters?



8. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find the side length x.

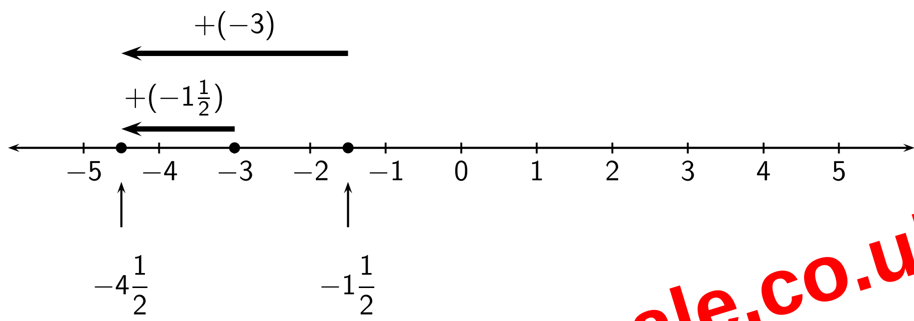


9. A 6 foot man casts an 8 foot shadow on the ground. How long is the shadow of a nearby 32 foot tree? Draw a figure involving similar triangles which illustrates the situation.

Solution. Both numbers are negative, so we move consistently left on the number line. We either start at $-1\frac{1}{2}$ and move 3 steps left, arriving at

$$\left(-1\frac{1}{2}\right) + (-3) = -\left(1\frac{1}{2} + 3\right) = -\left(\frac{3}{2} + \frac{6}{2}\right) = -\frac{9}{2} = -4\frac{1}{2}$$

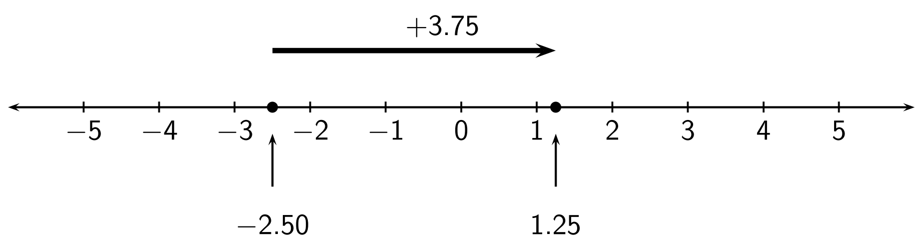
(upper arrow in the picture below); or we start at -3 and move $1\frac{1}{2}$ steps left (lower arrow). Either way, we arrive at $-4\frac{1}{2}$.



□

Example 153. Perform the signed number addition $-2.50 + 3.75$.

Solution. Starting at -2.50, we move a distance of 3.75 to the right, ending at 1.25, as shown.



□

In the last example, the *absolute value* of the sum was the *difference* obtained by subtracting the smaller absolute value from the larger ($3.75 - 2.50 = 1.25$), while the *sign* of the sum (+) was the same as the sign of the number with the larger absolute value. This is no accident, but illustrates a general rule about adding numbers with opposite signs.

Solution. The numbers have opposite signs, and the number with the larger absolute value is negative, so the sum will be negative. The absolute value of the sum is the difference in the individual absolute values,

$$
\begin{aligned}
&= 7\frac{1}{3} - 4\frac{3}{5} \\
&= 7\frac{5}{15} - 4\frac{9}{15} && \text{(using the LCD} = 15\text{)} \\
&= 6\frac{20}{15} - 4\frac{9}{15} && \text{(borrowing } 1 = \frac{15}{15} \text{ from 7)} \\
&= 2\frac{11}{15}.
\end{aligned}
$$

Finally, recalling that the sign of the sum is negative, we have

$$4\frac{3}{5} + \left(-7\frac{1}{3}\right) = -2\frac{11}{15}.$$

□

5.1.1 Exercises

Add.

1. $-6 + 19$
2. $12 + (-4)$
3. $-34 + (-28)$
4. $266 + (-265)$
5. $5\frac{3}{5} + \left(-4\frac{1}{2}\right)$
6. $-13.38 + (-9.03)$
7. $-1001.36 + 909$
8. $\left(-\frac{3}{4}\right) + 2$
9. $\left(-\frac{5}{6}\right) + (-5)$
10. $\left(3\frac{1}{5}\right) + \left(2\frac{5}{8}\right)$

Use an appropriate signed number addition for the following.

11. Find the temperature at noon in Anchorage if the temperature at dawn was $-11°$ F and the temperature subsequently rose by $36°$ F.
12. Find the height (in feet above ground level) of an elevator which started 30 feet below ground level and subsequently rose 70 feet.

Solution. $-32-(-15)=-32+(-(-15))=-32+15$. Applying the rule for adding signed numbers with opposite signs,

$$-32-(-15)=-32+15=-(32-15)=-17.$$

□

Example 167. Subtract $2\frac{7}{8}$ from $1\frac{5}{6}$.

Solution. Adding the opposite of $2\frac{7}{8}$ to $1\frac{5}{6}$, we get

$$\begin{aligned}1\frac{5}{6}+\left(-2\frac{7}{8}\right)&=-\left(2\frac{7}{8}-1\frac{5}{6}\right)\\&=-\left(2\frac{21}{24}-1\frac{20}{24}\right)\qquad(\text{LCD}=24)\\&=-1\frac{1}{24}.\end{aligned}$$

□

Example 168. Perform the subtraction $3.359-10.08$.

Solution. $3.359-10.08=3.359+(-10.08)=-(10.08-3.359)$. We perform the subtraction of absolute values vertically.

$$\begin{array}{r}10.080\\-\ \ 3.359\\\hline 6.721\end{array}$$

Remembering that the sign was negative, $3.359-10.08=-6.721$. □

Formerly "impossible" subtractions, such as

$$7-12$$

can now be easily performed.

$$7-12=7+(-12)=-(12-7)=-5.$$

Examples like this show that subtraction is **not commutative**: *changing the order of subtraction changes the result to its opposite.* In general, for any two signed numbers,

$$B-A=-(A-B).$$

Example 169. $22-100=-78=-(100-22)$.

We often need to find the *difference* of two unequal quantities. By convention, difference is given as a positive quantity. If the two quantities are A and B, their difference is either $A-B$ or $B-A$ (whichever is positive). Intuitively,

$$\text{difference}=\text{larger}-\text{smaller}.$$

Formally, the difference of A and B is defined to be the *absolute value* of $A-B$.

13. $\left(-1\frac{6}{7}\right)\left(-1\frac{1}{2}\right)$

14. $(-1.62)(1000)$

15. $1(-1)$

16. $(-3)(-50)(-2)$

17. $(-3)(0.5)(-0.7)(1)$

18. $(3)(-10)(2)(-5)$

19. $(-6)(-5)(-4)(-3)(0)$

20. $\left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{3}{8}\right)\left(\frac{4}{5}\right)$

5.4 Dividing Signed Numbers

Recall that two numbers are *reciprocal* if their product is 1. We extend this definition unchanged to signed numbers. For example, since

$$(-2) \times \left(-\frac{1}{2}\right) = 1,$$

-2 and $-\frac{1}{2}$ are reciprocal. Similarly, $-\frac{3}{8}$ and $-\frac{8}{3}$ are reciprocal, since

$$\left(-\frac{3}{8}\right)\left(-\frac{8}{3}\right) = 1.$$

Division by a signed number $N \neq 0$ can then be defined (as with positive numbers) as *multiplication by the reciprocal* of N, i.e.,

$$M \div N = M \cdot \frac{1}{N}$$

for any number M. Since a number and its reciprocal must have the same sign (why?), it follows that the rules for dividing signed numbers are exactly analogous to the rules for multiplying them.

> When two signed numbers are **divided** (and the divisor is nonzero)
>
> - if the dividend and divisor have the **same** sign, the quotient is positive
> - if the dividend and the divisor have **opposite** signs, the quotient is negative.
>
> In both cases, the absolute value of the quotient is the quotient of the individual absolute values.

5.4.2 Exercises

Perform the divisions, or state that they are undefined.

1. $(-24) \div (-8)$
2. $(-24) \div 8$
3. $66 \div 0$
4. $30 \div (-6)$
5. $(-30) \div 6$
6. $\dfrac{-26}{13}$
7. $\dfrac{-19}{0}$
8. $\dfrac{-72}{-18}$
9. $\dfrac{9.5}{-1.9}$
10. $601.03 \div (-1000)$
11. $0 \div (-1000)$
12. $\left(\frac{4}{5}\right) \div \left(-\frac{25}{32}\right)$
13. $\left(-4\frac{1}{2}\right) \div \left(-1\frac{7}{8}\right)$
14. $-10\frac{3}{4} \div 5$
15. $100 \div \frac{1}{4}$
16. $-110.98 \div 4$
17. $.5 \div .4$
18. $50 \div (-.4)$

5.5 Powers of Signed Numbers

Since exponents indicate repeated multiplication, there is no problem applying exponents to signed numbers.

$$(-4)^3 = (-4)(-4)(-4) = -64 \qquad \text{and} \quad (-2)^4 = (-2)(-2)(-2)(-2) = 16.$$

5.8.1 Exercises

1. Find the area of a rectangle whose length is 4.8 meters and whose width is 3.6 meters. Use the formula $A = lw$, where A is the area, l is the length, and w is the width.

2. Find the perimeter of the rectangle in the preceding exercise, using the formula $P = 2l + 2w$, where P is the perimeter and l, w are the length and width, respectively.

3. Find the length of the hypotenuse of a right triangle whose legs are 0.3 yards and 0.4 yards. Use the Pythagorean theorem.

4. How far does an object fall in 3 seconds? Use the formula $s = 16t^2$, where s is the distance fallen (in ft), and t is the time (in sec).

5. If a thermometer reads 22°C, find the temperature in °F. Use the formula $F = \frac{9}{5}C + 32$.

6. Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, gives the area ($A$) of a triangle (not necessarily a right triangle) with side lengths a, b and c, and semi-perimeter s. Use Heron's formula and the semi-perimeter formula $s = \frac{1}{2}(a + b + c)$ to find the area of a triangle with $a = b = c = 2$ ft.

7. The formula $A = P(1 + r)^t$ gives the amount A of money in a bank account t years after an initial amount P is deposited, when the annual interest rate is r. Find A after 2 years if the interest rate is 5% ($r = .05$), and the initial deposit was $P = \$500$.

8. Find the child's dosage for a 10-year old, if the adult dosage is 2.2 grams. Use the formula $C = \frac{t}{t+12} \cdot A$, where C is the child's dosage, t is the child's age, and A is the adult dosage.

5.9 Linear Equations in One Variable

An equation is a statement that two mathematical expressions are equal. If the statement has just one variable, and if

- the variable does not appear in the denominator of a fraction,
- the variable does not appear under a $\sqrt{\ }$ symbol,
- the variable is not raised to a power other than 1,

then we have a **linear equation in one variable**. Here are some examples of linear equations in one variable:

$$2x + 4 = 8 \qquad -3y = 12 \qquad z - 9 = -1 \qquad 2 - \frac{2}{3}t = 0.$$

A **solution** to an equation in one variable is a number which, when substituted for the variable, makes a true statement.

Example 210. Show that -4 is a solution to the equation $-3y = 12$

Solution. When we substitute -4 for y in the equation, we get

$$\begin{aligned} -3(-4) &= 12 \\ 12 &= 12, \qquad \text{a } \textit{true} \text{ statement.} \end{aligned}$$

□