3. ABC is a triangle with a right angle at C. If the median on the side a is the geometric mean of the sides b and c, show that $c = 3b$.

4. (a) Suppose $c = a + kb$ for a right triangle with legs a, b, and hypotenuse c. Show that $0 < k < 1$, and

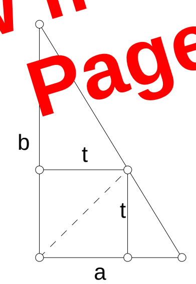
$$a : b : c = 1 - k^2 : 2k : 1 + k^2.$$

(b) Find two right triangles which are not similar, each satisfying $c = \frac{3}{4}a + \frac{4}{5}b$. [1]

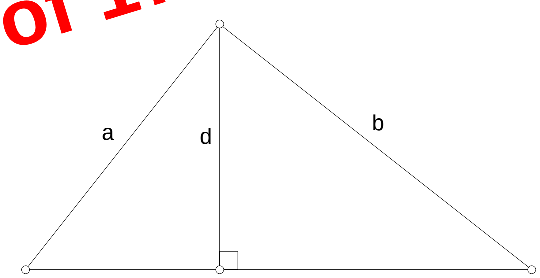
5. ABC is a triangle with a right angle at C. If the median on the side c is the geometric mean of the sides a and b, show that one of the acute angles is $15°$.

6. Let ABC be a right triangle with a right angle at vertex C. Let $CXPY$ be a square with P on the hypotenuse, and X, Y on the sides. Show that the length t of a side of this square is given by

$$\frac{1}{t} = \frac{1}{a} + \frac{1}{b}.$$



1/a + 1/b = 1/t.



1/a^2 + 1/b^2 = 1/d^2.

[1] $a : b : c = 12 : 35 : 37$ or $12 : 5 : 13$. More generally, for $h \leq k$, there is, up to similarity, a unique right triangle satisfying $c = ha + kb$ provided

(i) $h < 1 \leq k$, or

(ii) $\frac{\sqrt{2}}{2} \leq h = k < 1$, or

(iii) $h, k > 0$, $h^2 + k^2 = 1$.

There are two such right triangles if

$$0 < h < k < 1, \qquad h^2 + k^2 > 1.$$

7. Let ABC be a right triangle with sides a, b and hypotenuse c. If d is the height of on the hypotenuse, show that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{d^2}.$$

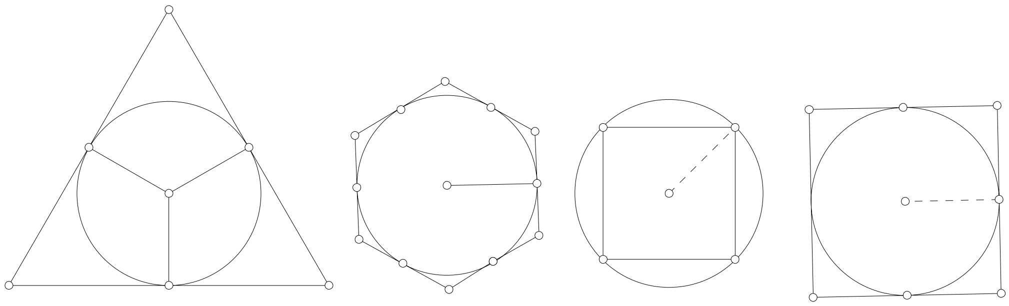
8. (Construction of integer right triangles) It is known that every right triangle of integer sides (without common divisor) can be obtained by choosing two relatively prime positive integers m and n, one odd, one even, and setting

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2.$$

(a) Verify that $a^2 + b^2 = c^2$.

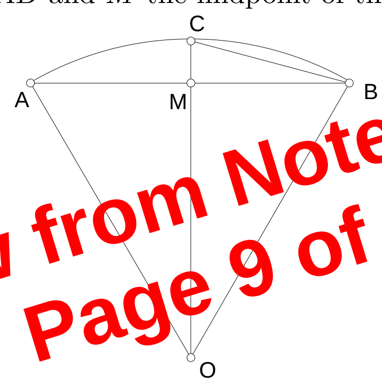
(b) Complete the following table to find *all* such right triangles with sides < 100:

	m	n	$a = m^2 - n^2$	$b = 2mn$	$c = m^2 + n^2$
(i)	2	1	3	4	5
(ii)	3	2			
(iii)	4	1			
(iv)	4	3			
(v)	5	2			
(vi)	5	[illegible]			
(vii)	6	1			
$(viii)$	6	5			
(ix)	7	2			
(x)	7	4			
(xi)	7	6			
(xii)	8	1			
$(xiii)$	8	3			
(xiv)	8	5			
(xv)	9	2			
(xvi)	9	4	65	72	97



Exercise

1. AB is a chord of length 2 in a circle $O(2)$. C is the midpoint of the minor arc AB and M the midpoint of the chord AB.



Show that (i) $CM = 2 - \sqrt{3}$; (ii) $BC = \sqrt{6} - \sqrt{2}$.

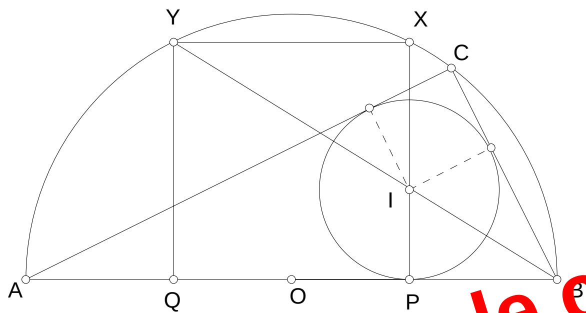
Deduce that

$$\tan 15^\circ = 2-\sqrt{3}, \qquad \sin 15^\circ = \frac{1}{4}(\sqrt{6}-\sqrt{2}), \qquad \cos 15^\circ = \frac{1}{4}(\sqrt{6}+\sqrt{2}).$$

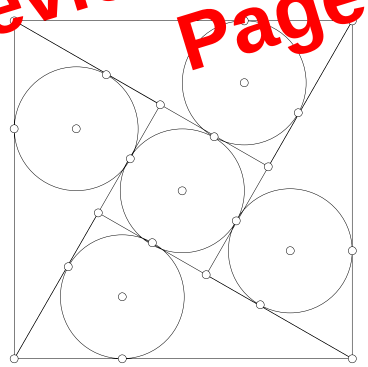
(a) Show that the right triangle ABC has the same area as the square $PXYQ$.

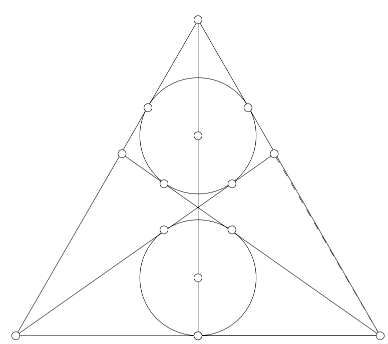
(b) Find the inradius of the triangle ABC. [5]

(c) Show that the incenter of $\triangle ABC$ is the intersection of PX and BY.



13. A square of side a is partitioned into 4 congruent right triangles and a small square, all with equal inradii r. Calculate r.





14. An equilateral triangle of side $2a$ is partitioned symmetrically into a quadrilateral, an isosceles triangle, and two other congruent triangles. If the inradii of the quadrilateral and the isosceles triangle are equal,

[5] $r = (3 - \sqrt{5})a$.

Proof. (1) The midpoint M of the segment II_A is on the circumcircle.

(2) The midpoint M' of I_BI_C is also on the circumcircle.

(3) MM' is indeed a diameter of the circumcircle, so that $MM' = 2R$.

(4) If D is the midpoint of BC, then $DM' = \frac{1}{2}(r_b + r_c)$.

(5) Since D is the midpoint of XX', $QX' = IX = r$, and $I_AQ = r_a - r$.

(6) Since M is the midpoint of II_A, MD is parallel to I_AQ and is half in length. Thus, $MD = \frac{1}{2}(r_a - r)$.

(7) It now follows from $MM' = 2R$ that $r_a + r_b + r_c - r = 4R$.

3. (a) Let ABC be an isosceles triangle with $a = 2$ and $b = c = 9$. Show that there is a circle with center I tangent to each of the excircles of triangle ABC.

   (b) Suppose there is a circle with center I tangent *externally* to each of the excircles. Show that the triangle is equilateral.

   (c) Suppose there is a circle with center I tangent *internally* to each of the excircles. Show that the triangle is equilateral.

4. Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths are proportional to its medians lengths in some order.

3.4 Power of a point with respect to a circle

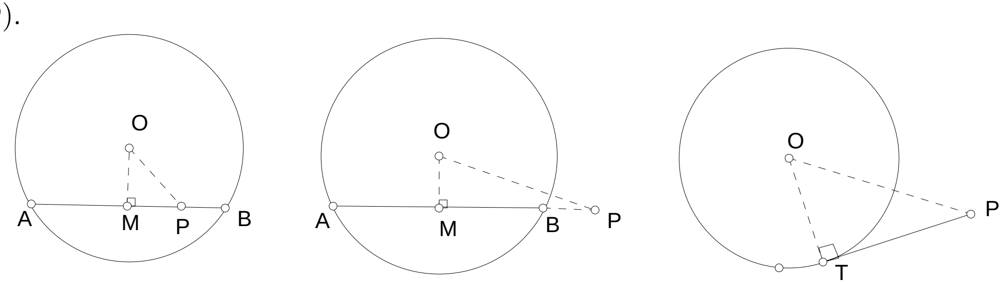
The *power* of a point P with respect to a circle $O(r)$ is defined as

$$O(r)_P := OP^2 - r^2.$$

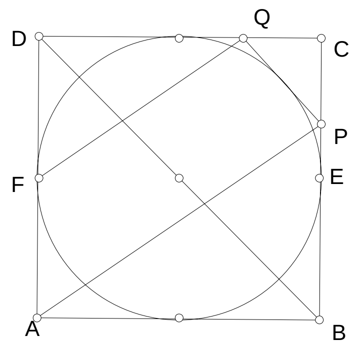
This number is positive, zero, or negative according as P is outside, on, or inside the circle.

3.4.1

For any line ℓ through P intersecting a circle (O) at A and B, the signed product $PA \cdot PB$ is equal to $(O)_P$, the power of P with respect to the circle (O).



If P is outside the circle, $(O)_P$ is the square of the tangent from P to (O).

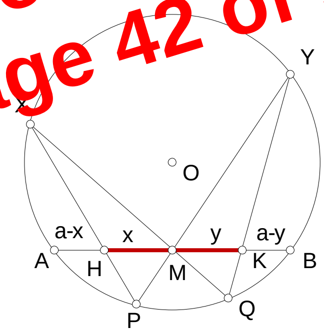


3. (The butterfly theorem) Let M be the midpoint of a chord AB of a circle (O). PY and QX are two chords through M. PX and QY intersect the chord AB at H and K respectively.

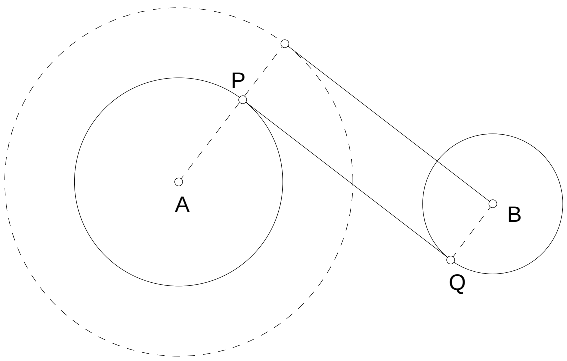
   (i) Use the sine formula to show that

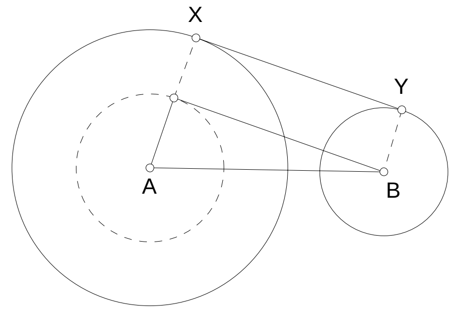
$$\frac{HX \cdot HP}{HM^2} = \frac{KY \cdot KQ}{KM^2}.$$

   (ii) Use the intersecting chords theorem to deduce that $HM = KM$.



4. P and Q are two points on the diameter AB of a semicircle. $K(T)$ is the circle tangent to the semicircle and the perpendiculars to AB at P and Q. Show that the distance from K to AB is the geometric mean of the lengths of AP and BQ.



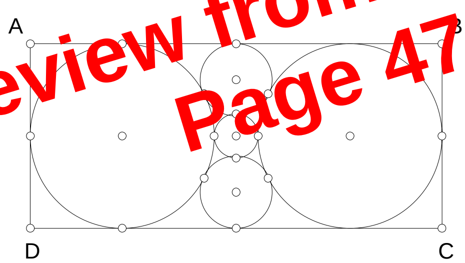


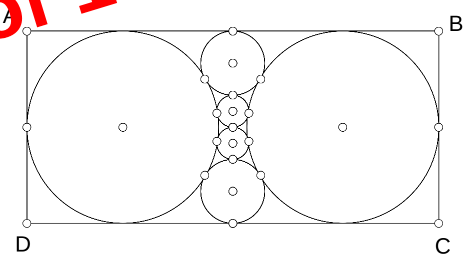
Suppose $d > |a - b|$ so that none of the circle contains the other. The external common tangent XY has length

$$\sqrt{d^2 - (a-b)^2}.$$

Exercise

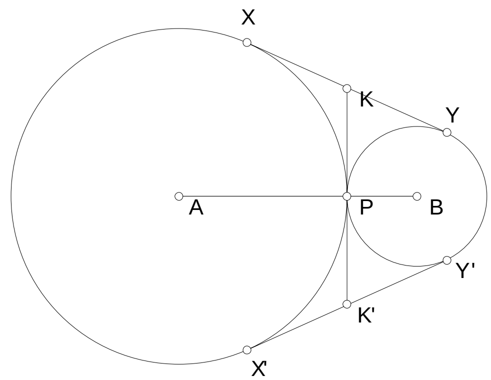
1. In each of the following cases, find the ratio $AB : BC$. [1]



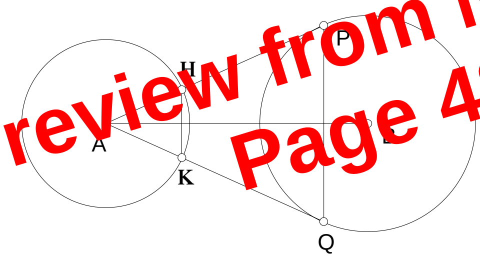


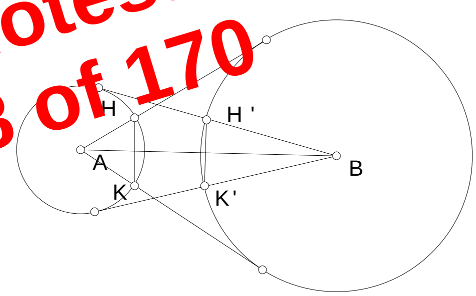
2. Two circles $A(a)$ and $B(b)$ are tangent externally at a point P. The common tangent at P intersects the two external common tangents XY, $X'Y'$ at K, K' respectively.
   (a) Show that $\angle AKB$ is a right angle.
   (b) What is the length PK?
   (c) Find the lengths of the common tangents XY and KK'.

[1] $\sqrt{3} : \sqrt{3} + 2$ in the case of 4 circles.



3. $A(a)$ and $B(b)$ are two circles with their centers at a distance d apart. AP and AQ are the tangents from A to circle $B(b)$. These tangents intersect the circle $A(a)$ at H and K. Calculate the length of HK in terms of d, a, and b. [2]



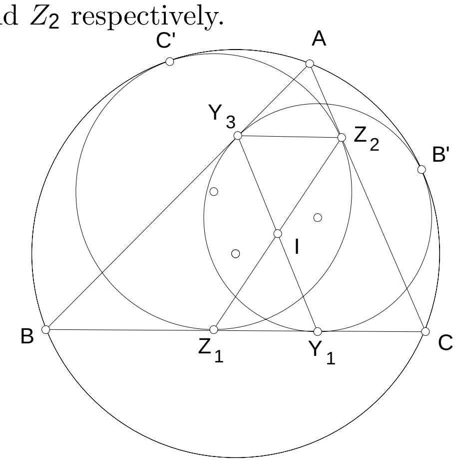


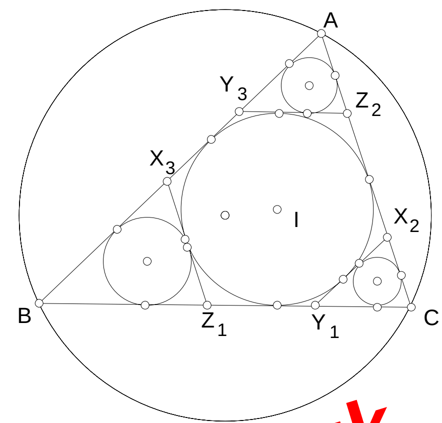
4. Tangents are drawn from the center of two given circles to the other circles. Show that the chords HK and $H'K'$ intercepted by the tangents are equal.

5. $A(a)$ and $B(b)$ are two circles with their centers at a distance d apart. From the extremity A' of the diameter of $A(a)$ on the line AB, tangents are constructed to the circle $B(b)$. Calculate the radius of the circle tangent internally to $A(a)$ and to these tangent lines. [3]

[2] Answer: $\frac{2ab}{d}$.
[3] Answer: $\frac{2ab}{d+a+b}$.

and Y_3 respectively, and that in angle C touch the sides BC and AC at Z_1 and Z_2 respectively.





Each of the segments X_2X_3, Y_3Y_1, and Z_1Z_2 has the incenter I as midpoint. It follows that the triangles IY_1Z_1 and IY_3Z_2 are congruent, and the segment Y_3Z_2 is parallel to the side BC containing the segment Y_1Z_1, and is tangent to the incircle. Therefore, the triangles AY_3Z_2 and ABC are similar, the ratio of similarity being

$$\frac{Y_3Z_2}{a} = \frac{h_a - 2r}{h_a},$$

with $h_a = \frac{2\triangle}{a} = \frac{2rs}{a}$, the altitude of triangle ABC on the side BC. Simplifying this, we obtain $\frac{Y_3Z_2}{a} = \frac{s-a}{s}$. From this, the inradius of the triangle AY_3Z_2 is given by $r_a = \frac{s-a}{s} \cdot r$. Similarly, the inradii of the triangles BZ_1X_3 and CX_2BY_1 are $r_b = \frac{s-b}{s} \cdot r$ and $r_c = \frac{s-c}{s} \cdot r$ respectively. From this, we have

$$r_a + r_b + r_c = r.$$

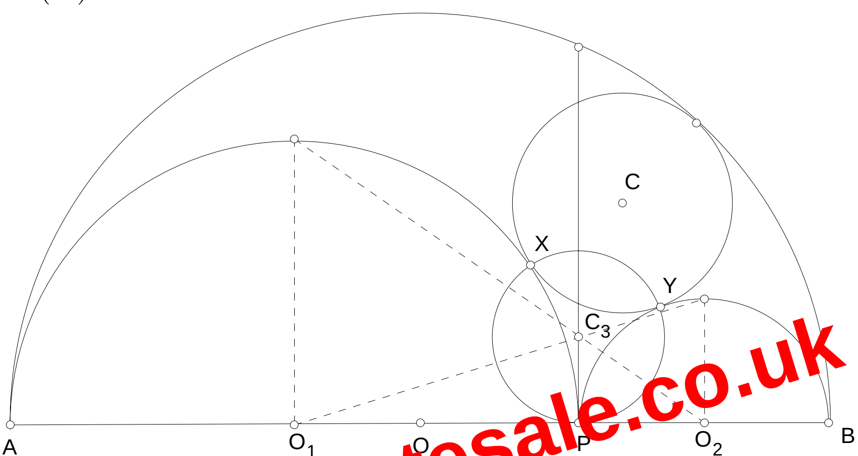
We summarize this in the following proposition.

Proposition

If tangents to the incircles of a triangle are drawn parallel to the sides, cutting out three triangles each similar to the given one, the sum of the inradii of the three triangles is equal to the inradius of the given triangle.

5.1.5 Construction of incircle of shoemaker's knife

Locate the point C_3 as in §??. Construct circle $C_3(P)$ to intersect $O_1(a)$ and $O_2(b)$ at X and Y respectively. Let the lines O_1X and O_2Y intersect at C. Then $C(X)$ is the incircle of the shoemaker's knife.



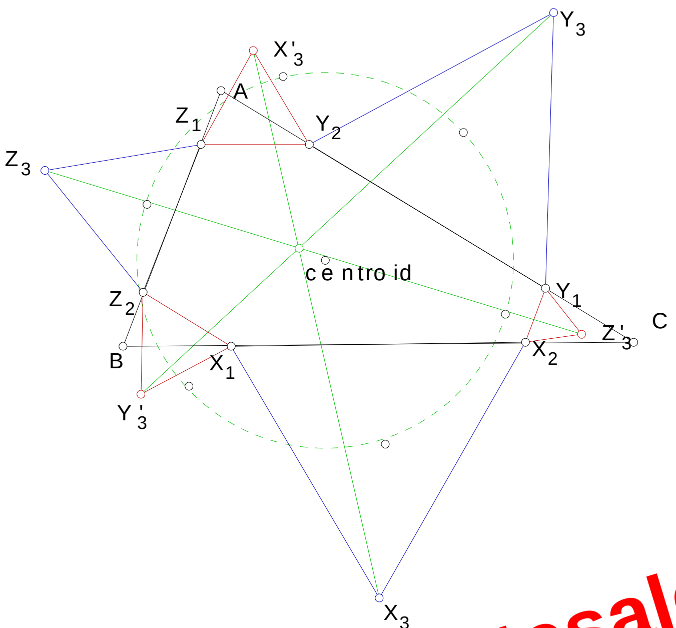
Note that $C_3(P)$ is the Bankoff circle, which has the same radius as the Archimedean circles.

Exercise

1. Show that the area of triangle CO_1O_2 is
$$\frac{ab(a+b)^2}{a^2+ab+b^2}.$$

2. Show that the center C of the incircle of the shoemaker's knife is at a distance 2ρ from the line AB.

3. Show that the area of the shoemaker's knife to that of the heart (bounded by semicircles $O_1(a)$, $O_2(b)$ and the *lower* semicircle $O(a+b)$) is as ρ to $a+b$.

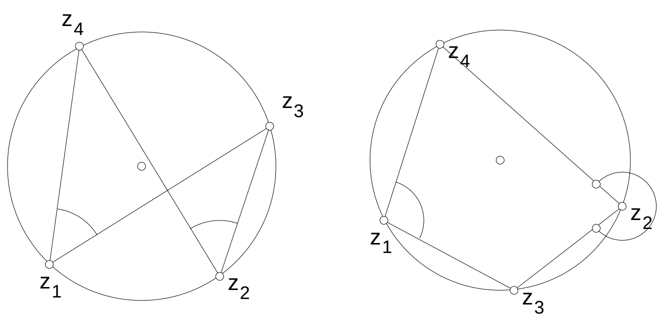


6.5 Concyclic points

Four non-collinear points z_1, z_2, z_3, z_4 are concyclic if and only if the *cross ratio*

$$(z_1, z_2; z_3, z_4) := \frac{z_4 - z_1}{z_3 - z_1} \Big/ \frac{z_4 - z_2}{z_3 - z_2} = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}$$

is a real number.



and are constructible, since y_1 is constructible. Similarly, if we write

$$z_3 = \omega^3 + \omega^5 + \omega^{14} + \omega^{12}, \qquad z_4 = \omega^{10} + \omega^{11} + \omega^7 + \omega^6,$$

we find that $z_3 + z_4 = y_2$, and $z_3z_4 = \omega + \omega^2 + \cdots + \omega^{16} = -1$, so that z_3 and z_4 are the roots of the quadratic equation

$$z^2 - y_2z - 1 = 0$$

and are also constructible.

Finally, further separating the terms of z_1 into two pairs, by putting

$$t_1 = \omega + \omega^{16}, \qquad t_2 = \omega^{13} + \omega^4,$$

we obtain

$$\begin{aligned} t_1 + t_2 &= z_1, \\ t_1t_2 &= (\omega + \omega^{16})(\omega^{13} + \omega^4) = \omega^{14} + \omega^5 + \omega^{12} + \omega^3 = z_3. \end{aligned}$$

It follows that t_1 and t_2 are the roots of the quadratic equation

$$t^2 - z_1t + z_3 = 0,$$

and are constructible, since z_1 and z_3 are constructible.

6.6.2 Explicit construction of a regular 17-gon [4]

To construct two vertices of the regular 17-gon inscribed in a given circle $O(A)$.

1. On the radius OB perpendicular to OA, mark a point J such that $OJ = \frac{1}{4}OA$.

2. Mark a point E on the segment OA such that $\angle OJE = \frac{1}{4}\angle OJA$.

3. Mark a point F on the diameter through A such that O is between E and F and $\angle EJF = 45°$.

4. With AF as diameter, construct a circle intersecting the radius OB at K.

[4] H.S.M.Coxeter, Introduction to Geometry, 2nd ed. p.27.

For every point P (except the midpoint of AB), let P' be the point on AC such that $PP' \perp AB$.

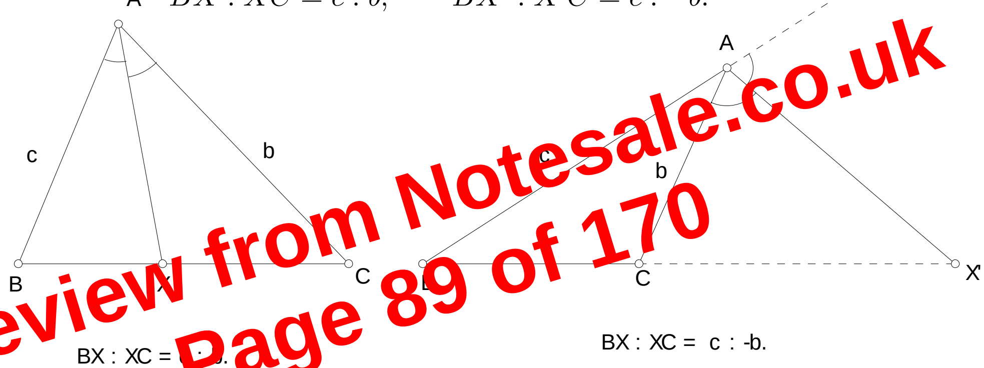
The intersection Q of the lines $P'M$ and AB is the harmonic conjugate of P with respect to AB.

7.2 Apollonius Circle

7.2.1 Angle bisector Theorem

If the internal (repsectively external) bisector of angle BAC intersect the line BC at X (respectively X'), then

$$BX : XC = c : b; \qquad BX' : X'C = c : -b.$$



7.2.2 Example

The points X and X' are harmonic conjugates with respect to BC, since

$$BX : XC = c : b, \quad \text{and} \quad BX' : X'C = c : -b.$$

7.2.3

A and B are two fixed points. For a given positive number $k \neq 1$, [1] the locus of points P satisfying $AP : PB = k : 1$ is the circle with diameter XY, where X and Y are points on the line AB such that $AX : XB = k : 1$ and $AY : YB = k : -1$.

[1] If $k - 1$, the locus is clearly the perpendicular bisector of the segment AB.

Proof. ($\Longrightarrow$) Let W be the point on AB such that $CW//XY$. Then,

$$\frac{BX}{XC} = \frac{BZ}{ZW}, \quad \text{and} \quad \frac{CY}{YA} = \frac{WZ}{ZA}.$$

It follows that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{BZ}{ZW} \cdot \frac{WZ}{ZA} \cdot \frac{AZ}{ZB} = \frac{BZ}{ZB} \cdot \frac{WZ}{ZW} \cdot \frac{AZ}{ZA} = (-1)(-1)(-1) = -1.$$

($\Longleftarrow$) Suppose the line joining X and Z intersects AC at Y'. From above,

$$\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = -1 = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}.$$

It follows that

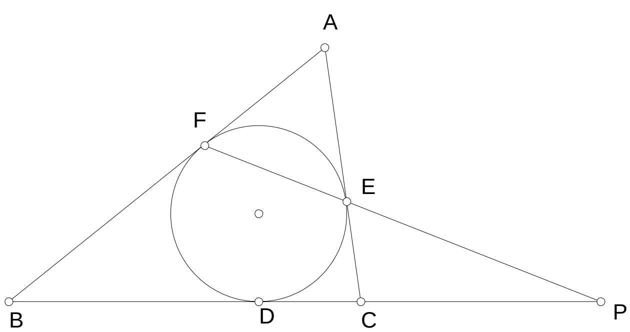
$$\frac{CY'}{Y'A} = \frac{CY}{YA}.$$

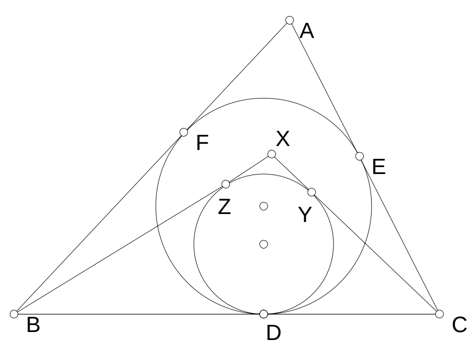
The points Y' and Y divide the segment CA in the same ratio. These must be the same point, and X, Y, Z are collinear.

Exercise

1. M is a point on the median AD of $\triangle ABC$ such that $AM : MD = p : q$. The line CM intersects the side AB at N. Find the ratio $AN : NB$. [3]

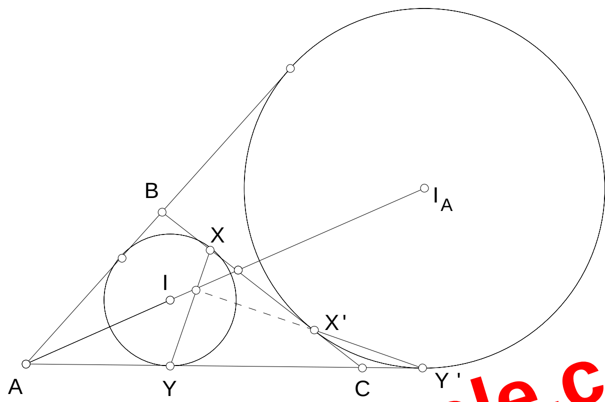
2. The incircle of $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. Suppose $AB \neq AC$. The line joining E and F meets BC at P. Show that P and D divide BC harmonically.





[3] Answer: $AN : NB = p : 2q$.

3. The incircle of $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D also, and touches CX and XB at Y and Z respectively. Show that E, F, Z, Y are concyclic. [4]



4. Given a triangle ABC, let the incircle and the ex-circle on BC touch the side BC at X and X' respectively, and the line AC at Y and Y' respectively. Then the lines XY and $X'Y'$ intersect on the bisector of angle A, at the projection of B on this bisector.

7.4 The Ceva Theorem

Let X, Y, Z be points on the lines BC, CA, AB respectively. The lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

Proof. ($\Longrightarrow$) Suppose the lines AX, BY, CZ intersect at a point P. Consider the line BPY cutting the sides of $\triangle CAX$. By Menelaus' theorem,

$$\frac{CY}{YA} \cdot \frac{AP}{PX} \cdot \frac{XB}{BC} = -1, \quad \text{or} \quad \frac{CY}{YA} \cdot \frac{PA}{XP} \cdot \frac{BX}{BC} = +1.$$

[4] IMO 1996.

Exercise

1. If $X = yB + zC$, then the isotomic conjugate is $X' = zB + yC$.

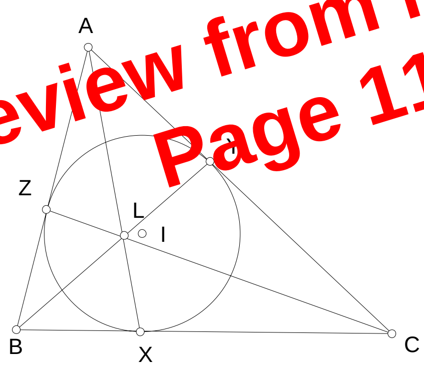
2. X', Y', Z' are collinear if and only if X, Y, Z are collinear.

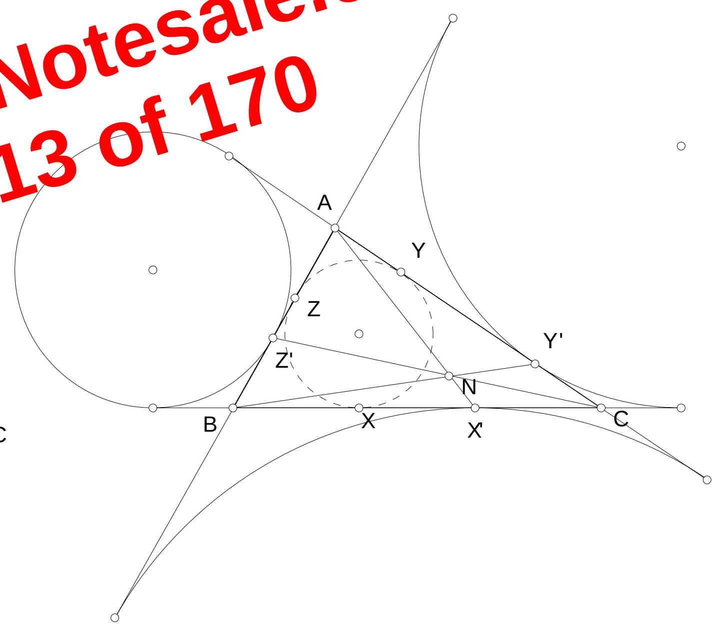
8.5.2 Gergonne and Nagel points

Suppose the incircle $I(r)$ of triangle ABC touches the sides BC, CA, and AB at the points X, Y, and Z respectively.

$$\begin{array}{ccccccccc} & & BX & : & XC & = & & s-b & : & s-c, \\ AY & : & & & YC & = & s-a & : & & s-c, \\ AZ & : & ZB & & & = & s-a & : & s-b & . \end{array}$$

This means the cevians AX, BY, CZ are concurrent. The intersection is called the Gergonne point of the triangle, sometimes also known as the Gergonne point.





Let X', Y', Z' be the isotomic conjugates of X, Y, Z on the respective sides. The point X' is indeed the point of contact of the excircle $I_A(r_1)$ with the side BC; similarly for Y' and Z'. The cevians AX', BY', CZ' are

7. The Gergonne point of the triangle $K_AK_BK_C$ is the symmedian point K of $\triangle ABC$.

8. Characterize the triangles of which the midpoints of the altitudes are collinear. [8]

9. Show that the *mirror image* of the orthocenter H in a side of a triangle lies on the circumcircle.

10. Let P be a point in the plane of $\triangle ABC$, G_A, G_B, G_C respectively the centroids of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$. Show that AG_A, BG_B, and CG_C are concurrent. [9]

11. If the sides of a triangle are in arithmetic progression, then the line joining the centroid to the incenter is parallel to a side of the triangle.

12. If the squares of a triangle are in arithmetic progression, then the line joining the centroid and the symmedian point is parallel to a side of the triangle.

8.6.8

In §? we have established, using the trigonometric version of Ceva theorem, the concurrency of the lines joining each vertex of a triangle to the point of contact of the circumcircle with the mixtilinear incircle in that angle. Suppose the line AA', BB', CC' intersects the sides BC, CA, AB at points X, Y, Z respectively. We have

$$\frac{BX}{XC} = \frac{c}{b}\cdot\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{(s-b)/b^2}{(s-c)/c^2}.$$

$$\begin{array}{ccccccccc}
 & & BX & : & XC & = & & \frac{s-b}{b^2} & : & \frac{s-c}{c^2}, \\
AY & : & & & YC & = & \frac{s-c}{a^2} & : & & \frac{s-c}{c^2}, \\
AZ & : & ZB & & & = & \frac{s-c}{a^2} & \frac{s-b}{b^2} & & .
\end{array}$$

---

[8] More generally, if P is a point with nonzero homogeneous coordinates with respect to $\triangle ABC$, and AP, BP, CP cut the opposite sides at X, Y and Z respectively, then the midpoints of AX, BY, CZ are never collinear. It follows that the orthocenter must be a vertex of the triangle, and the triangle must be right. See MG1197.844.S854.

[9] At the centroid of A, B, C, P; see MGQ781.914.

This latter equation can be rewritten as

$$\frac{c+x+ka}{k} = \frac{b+x+(1-k)a}{1-k}, \tag{9.1}$$

or

$$\frac{c+x}{k} = \frac{b+x}{1-k}, \tag{9.2}$$

from which

$$k = \frac{x+c}{2x+b+c}.$$

Now substitution into (1) gives

$$x^2(2x+b+c)^2 = (2x+b+c)[(x+c)b^2+(x+b)c^2] - (x+b)(x+c)a^2.$$

Rearranging, we have

$$\begin{aligned}
(x+b)(x+c)a^2 &= (2x+b+c)[(x+c)b^2+(x+b)c^2 - x^2[(x+b)+(x+c)]] \\
&= (2x+b+c)[(x+b)(c^2-x^2)+(x+c)(b^2-x^2)] \\
&= (2x+b+c)(x+b)(x+c)[(c-x)+(b-x)] \\
&= (2x+b+c)(x+b)(x+c)[(b+c)-2x] \\
&= (x+b)(x+c)[(b+c)^2-4x^2].
\end{aligned}$$

From this,

$$x^2 = \frac{1}{4}((b+c)^2-a^2) = \frac{1}{4}(b+c+a)(b+c-a) = s(s-a).$$

9.1.2

Lau [1] has proved an interesting formula which leads to a simple construction of the point P. If the angle between the median AD and the angle bisector AX is θ, then

$$m_a \cdot w_a \cdot \cos\theta = s(s-a).$$

[1] Solution to Crux 1097.

Exercise

1. Show that
$$r' = \frac{s - \sqrt{s(s-a)}}{a} \cdot r.$$

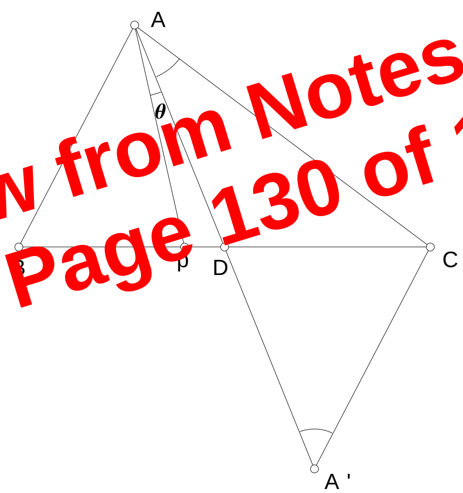
2. Show that the circle with XY as diameter intersects BC at P if and only if $\triangle ABC$ is isosceles. [2]

9.1.4 Proof of Lau's formula

Let θ be the angle between the median and the bisector of angle A.

Complete the triangle ABC into a parallelogram $ABA'C$. In triangle $AA'C$, we have

$$\begin{array}{rclrclrcl} AA' & = & 2m_a, & AC & = & b, & A'C & = & c; \\ \angle ACA' & = & 180^\circ - \alpha, & \angle AA'C & = & \frac{\alpha}{2} + \theta, & \angle A'AC & = & \frac{\alpha}{2} - \theta. \end{array}$$



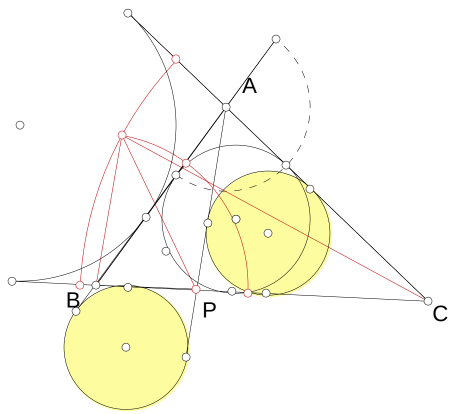
By the sine formula,

$$\frac{b+c}{2m_a} = \frac{\sin(\frac{\alpha}{2}+\theta) + \sin(\frac{\alpha}{2}-\theta)}{\sin(180^\circ - \alpha)} = \frac{2\sin\frac{\alpha}{2}\cos\theta}{\sin\alpha} = \frac{\cos\theta}{\cos\frac{\alpha}{2}}.$$

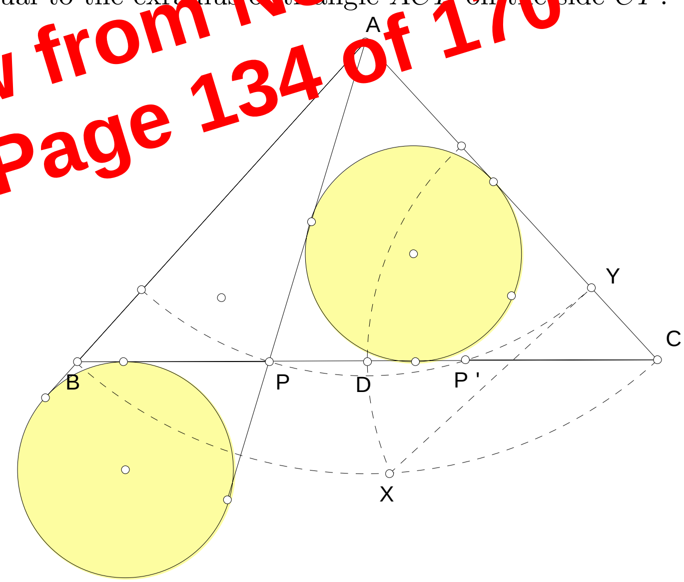
From this it follows that

$$m_a \cdot \cos\theta = \frac{b+c}{2} \cdot \cos\frac{\alpha}{2}.$$

[2]Hint: AP is tangent to the circle XYP.



4. Let ABC be an isosceles triangle, D the midpoint of the base BC. On the minor arc BC of the circle $A(B)$, mark a point X such that $CX = CD$. Let Y be the projection of X on the side AC. Let P be a point on BC such that $AP = AY$. Show that the inradius of triangle ABP is equal to the exradius of triangle ACP on the side CP.



spective from their common incenter I. The line joining their circumcenters passes through I. Note that T is the circumcenter of triangle $I_1I_2I_3$, the circumradius being the common radius t of the three circles. This means that T, O and I are collinear. Since

$$\frac{I_3I_1}{CA} = \frac{I_1I_2}{AB} = \frac{I_2I_3}{BC} = \frac{r-t}{r},$$

we have $t = \frac{r-t}{r} \cdot R$, or

$$\frac{t}{R} = \frac{r}{R+r}.$$

This means I divides the segment OT in the ratio

$$TI : IO = -r : R + r.$$

Equivalently, $OT : TI = R : r$, and T is the internal center of similitude of the circumcircle and the incircle.

9.3.2 Construction

Let O and I be the circumcenter and the incenter of triangle ABC.

(1) Construct the perpendicular from I to BC, intersecting the latter at X.

(2) Construct the perpendicular from O to BC, intersecting the circumcircle at M (so that IX and OM are directly parallel).

(3) Join OX and IM. Through their intersection P draw a line parallel to IX, intersecting OI at T, the internal center of similitude of the circumcircle and incircle.

(4) Construct the circle $T(P)$ to intersect the segments IA, IB, IC at I_1, I_2, I_3 respectively.

(5) The circles $I_j(T)$, $j = 1, 2, 3$ are three equal circles through T each tangent to two sides of the triangle.

Since $I = \frac{1}{2s}(a \cdot A + b \cdot B + c \cdot C)$, the homongeneous coordinates of I_1 with respect to ABC are

$$\begin{aligned} & a\cos\frac{\alpha}{2} : b\cos\frac{\alpha}{2} + 2s\sin\frac{\beta}{2} : c\cos\frac{\alpha}{2} + 2s\sin\frac{\gamma}{2} \\ = \; & a : b(1 + 2\cos\frac{\gamma}{2}) : c(1 + 2\cos\frac{\beta}{2}). \end{aligned}$$

Here, we have made use of the sine formula:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \frac{2s}{\sin\alpha + \sin\beta + \sin\gamma} = \frac{2s}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}.$$

Since I has homogeneous coordinates $a : b : c$, it is easy to see that the line II_1 intersects BC at the point A' with homogeneous coordinates

$$0 : b\cos\frac{\gamma}{2} : c\cos\frac{\beta}{2} = 0 : b\sec\frac{\beta}{2} : c\sec\frac{\gamma}{2}.$$

Similarly, B' and C' have coordinates

$$\begin{aligned} A' &= 0 : b\sec\frac{\beta}{2} : c\sec\frac{\gamma}{2}, \\ B' &= a\sec\frac{\alpha}{2} : 0 : c\sec\frac{\gamma}{2}, \\ C' &= a\sec\frac{\alpha}{2} : b\sec\frac{\beta}{2} : 0. \end{aligned}$$

From these, it is clear that AA', BB', CC' intersect at a point with homogeneous coordinates

$$a\sec\frac{\alpha}{2} : b\sec\frac{\beta}{2} : c\sec\frac{\gamma}{2}.$$

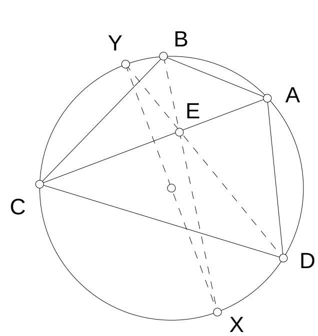
Exercise

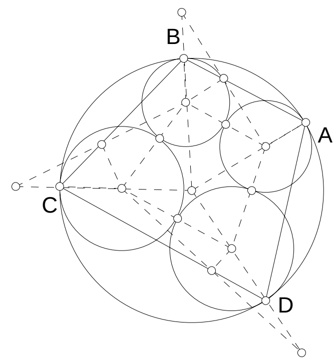
1. Let O_1, O_2, O_3 be the circumcenters of triangles I_1BC, I_2CA, I_3AB respectively. Are the lines O_1I_1, O_2I_2, O_3I_3 concurrent?

9.5 Malfatti circles

9.5.1 Construction Problem

Given a triangle, to construct three circles mutually tangent to each other, each touching two sides of the triangle.

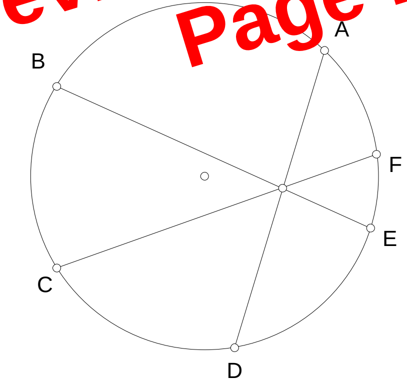


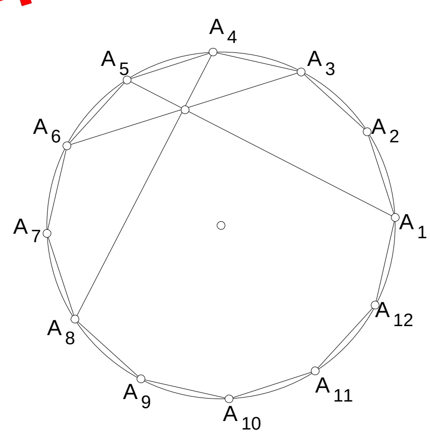


Beginning with any circle $K(A)$ tangent internally to $O(A)$, a chain of four circles can be completed to touch (O) at each of the four points A, B, C, D.

Exercise

1. Let A, B, C, D, E, F be six consecutive points on a circle. Show that the chords AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.





2. Let $A_1A_2 \dots A_{12}$ be a regular $12-$ gon. Show that the diagonals A_1A_5, A_3A_6 and A_4A_8 are concurrent.

3. Inside a given circle C is a chain of six circles C_i, $i = 1, 2, 3, 4, 5, 6,$

$$x^2 = c^2 + d^2 - 2cd\cos\delta.$$

Eliminating x, we have

$$a^2 + b^2 - c^2 - d^2 = 2ab\cos\beta - 2cd\cos\delta,$$

Denote by S the area of the quadrilateral. Clearly,

$$S = \frac{1}{2}ab\sin\beta + \frac{1}{2}cd\sin\delta.$$

Combining these two equations, we have

$$\begin{aligned}
& 16S^2 + (a^2+b^2-c^2-d^2)^2 \\
= & 4(ab\sin\beta + cd\sin\delta)^2 + 4(ab\cos\beta - cd\cos\delta)^2 \\
= & 4(a^2b^2 + c^2d^2) - 8abcd(\cos\beta\cos\delta - \sin\beta\sin\delta) \\
= & 4(a^2b^2 + c^2d^2) - 8abcd\cos(\beta+\delta) \\
= & 4(a^2b^2 + c^2d^2) - 8abcd\left[2\cos^2\frac{\beta+\delta}{2} - 1\right] \\
= & 4(ab+cd)^2 - 16abcd\cos^2\frac{\beta+\delta}{2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
16S^2 = & 4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 - 16abcd\cos^2\frac{\beta+\delta}{2} \\
= & [2(ab+cd) + (a^2+b^2-c^2-d^2)][2(ab+cd) - (a^2+b^2-c^2-d^2)] \\
& -16abcd\cos^2\frac{\beta+\delta}{2} \\
= & [(a+b)^2 - (c-d)^2][(c+d)^2 - (a-b)^2] - 16abcd\cos^2\frac{\beta+\delta}{2} \\
= & (a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b) \\
& -16abcd\cos^2\frac{\beta+\delta}{2}.
\end{aligned}$$

Writing

$$2s := a+b+c+d,$$

we reorganize this as

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\beta+\delta}{2}.$$

10.1.1 Cyclic quadrilateral

If the quadrilateral is *cyclic*, then $\beta + \delta = 180°$, and $\cos\frac{\beta+\delta}{2} = 0$. The area formula becomes

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{1}{2}(a+b+c+d)$.

Exercise

1. If the lengths of the sides of a quadrilateral are fixed, its area is greatest when the quadrilateral is cyclic.
2. Show that the Heron formula for the area of a triangle is a special case of this formula.

10.2 Ptolemy's Theorem

Suppose the quadrilateral $ABCD$ is cyclic. Then, $\beta + \delta = 180°$, and $\cos\beta = -\cos\delta$. It follows that

$$\frac{a^2+b^2-x^2}{2ab} + \frac{c^2+d^2-x^2}{2cd} = 0,$$

and

$$x^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Similarly, the other diagonal y is given by

$$y^2 = \frac{(ab+cd)(ac+bd)}{(ad+bc)}.$$

From these, we obtain

$$xy = ac + bd.$$

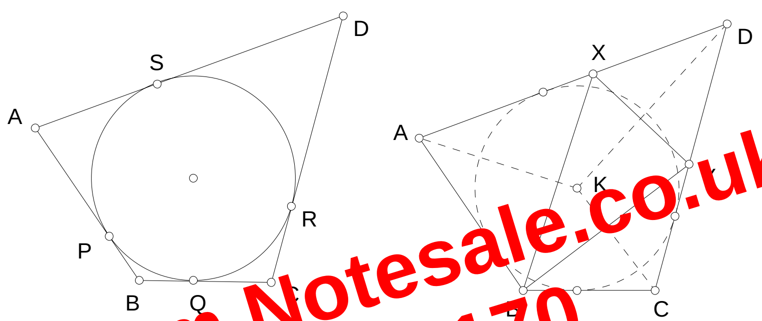
This is Ptolemy's Theorem. We give a synthetic proof of the theorem and its converse.

10.3 Circumscriptible quadrilaterals

A quadrilateral is said to be *circumscriptible* if it has an incircle.

10.3.1 Theorem

A quadrilateral is circumscriptible if and only if the two pairs of opposite sides have equal total lengths.
Proof. (Necessity) Clear.



(Sufficiency) Suppose $AB + CD = BC + DA$, and $AB < AD$. Then $BC < CD$, and there are points $\begin{matrix} X \\ Y \end{matrix}$ on $\begin{matrix} AD \\ CD \end{matrix}$ such that $\begin{matrix} AX = AB \\ CY = CD \end{matrix}$. Then $DX = DY$. Let K be the circumcircle of triangle BXY. AK bisects angle A since the triangles AKX and AKB are congruent. Similarly, CK and DK are bisectors of angles B and C respectively. It follows that K is equidistant from the sides of the quadrilateral. The quadrilateral admits of an incircle with center K.

10.3.2 [8]

Let $ABCD$ be a circumscriptible quadrilateral, X, Y, Z, W the points of contact of the incircle with the sides. The diagonals of the quadrilaterals $ABCD$ and $XYZW$ intersect at the same point.

[8] See Crux 199. This problem has a long history, and usually proved using projective geometry. Charles Trigg remarks that the Nov.-Dec. issue of Math. Magazine, 1962, contains nine proofs of this theorem. The proof here was given by Joseph Konhauser.

10.9.1

(a) If Q is cyclic, then $\mathsf{Q}_{(O)}$ is circumscriptible.

(b) If Q is circumscriptible, then $\mathsf{Q}_{(O)}$ is cyclic. [24]

(c) If Q is cyclic, then $\mathsf{Q}_{(I)}$ is a rectangle.

(d) If Q, is cyclic, then the nine-point circles of BCD, CDA, DAB, ABC have a point in common. [25].

Exercise

1. Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles. [26]

2. Let P be a quadrilateral inscribed in a circle (O) and let Q be the quadrilateral formed by the centers of the four circles internally touching (O) and each of the two diagonals of P. Then the incenters of the four triangles having for sides the sides and diagonals of P form a rectangle inscribed in Q. [27]

10.10

10.10.1

The diagonals of a quadrilateral $ABCD$ intersect at P. The orthocenters of the triangle PAB, PBC, PCD, PDA form a parallelogram that is similar to the figure formed by the centroids of these triangles. What is "centroids" is replaced by circumcenters? [28]

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[24] E1055.532.S538.(V.Thébault)

[25] Crux 2276

[26] E619.444.S451. (W.B.Clarke)

[27] Thébault, AMM 3887.38.S837. See editorial comment on 837.p486.

[28] Crux 1820.