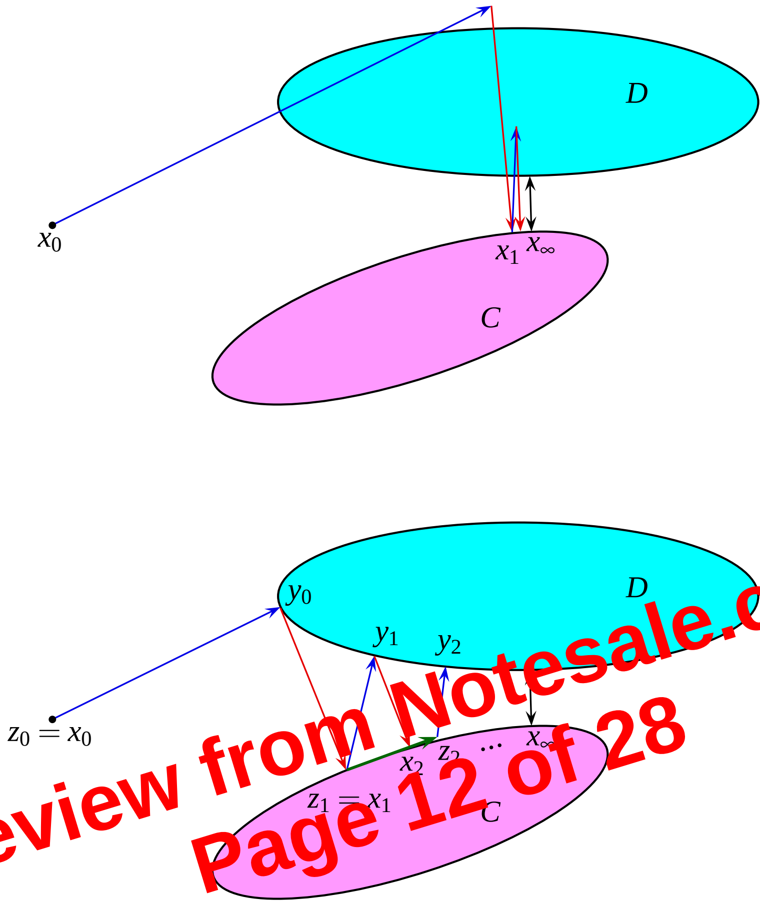

Fig. 10.2 Forward-backward versus Beck-Teboulle : As in Example 10.12, let C and D be two closed convex sets and consider the problem (10.30) of finding a point x_∞ in C at minimum distance from D. Let us set $f_1 = \iota_C$ and $f_2 = d_D^2/2$. Top: The forward–backward algorithm with $\gamma_n \equiv 1.9$ and $\lambda_n \equiv 1$. As seen in Example 10.12, it reduces to the alternating projection method (10.31). Bottom: The Beck-Teboulle algorithm.

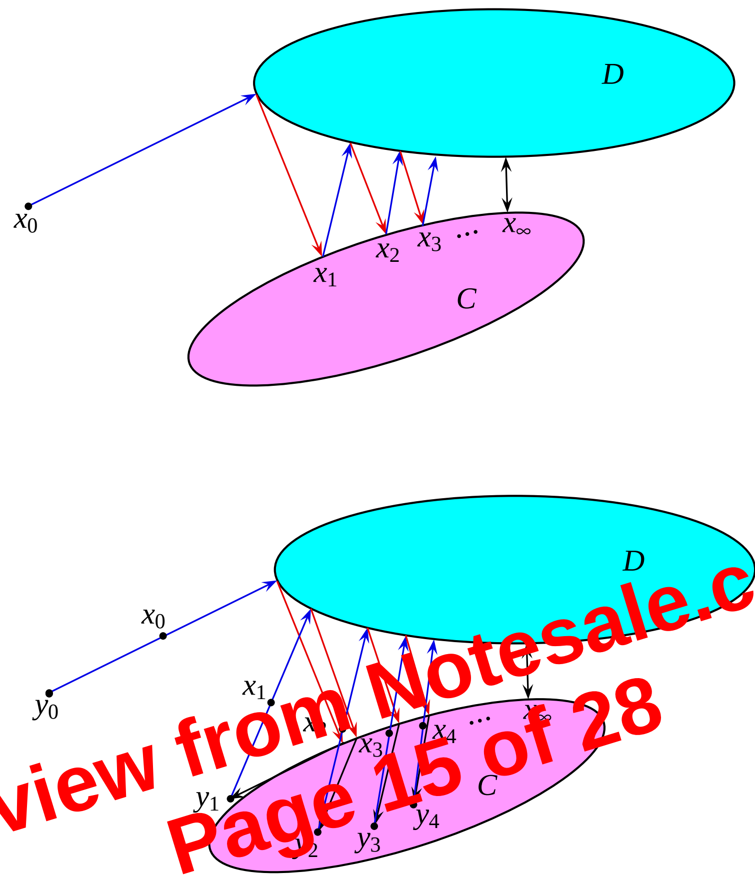

Fig. 10.3 Forward-backward versus Douglas–Rachford: As in Example 10.12, let C and D be two closed convex sets and consider the problem (10.30) of finding a point x_∞ in C at minimum distance from D. Let us set $f_1 = \iota_C$ and $f_2 = d_D^2/2$. Top: The forward–backward algorithm with $\gamma_n \equiv 1$ and $\lambda_n \equiv 1$. As seen in Example 10.12, it assumes the form of the alternating projection method (10.31). Bottom: The Douglas–Rachford algorithm with $\gamma = 1$ and $\lambda_n \equiv 1$. Table 10.1.xii yields $\mathrm{prox}_{f_1} = P_C$ and Table 10.1.vi yields $\mathrm{prox}_{f_2} \colon x \mapsto (x + P_D x)/2$. Therefore the updating rule in Algorithm 10.15 reduces to $x_n = (y_n + P_D y_n)/2$ and $y_{n+1} = P_C(2x_n - y_n) + y_n - x_n = P_C(P_D y_n) + y_n - x_n$.

(such techniques were introduced in [110, 111] and have been used in the context of convex feasibility problems in [10, 43, 45]). To this end, observe that (10.53) can be rewritten in $\mathscr{H}$ as

$$\underset{\substack{(x_1,\ldots,x_m)\in\mathscr{H}\\ x_1=\cdots=x_m}}{\text{minimize}}\; f_1(x_1)+\cdots+f_m(x_m). \tag{10.55}$$

If we denote by $x=(x_1,\ldots,x_m)$ a generic element in $\mathscr{H}$, (10.55) is equivalent to

$$\underset{x\in\mathscr{H}}{\text{minimize}}\; \iota_D(x)+f(x), \tag{10.56}$$

where

$$\begin{cases} D=\big\{(x,\ldots,x)\in\mathscr{H} \mid x\in\mathbb{R}^N\big\} \\ f\colon x\mapsto f_1(x_1)+\cdots+f_m(x_m). \end{cases} \tag{10.57}$$

We are thus back to a problem involving two functions in the larger space $\mathscr{H}$. In some cases, this observation makes it possible to obtain convergent methods from the algorithms discussed in the preceding sections. For instance, the following parallel algorithm was derived from the Douglas–Rachford algorithm in [54] (see also [49] for further analysis and connections with Spingarn's splitting method [120]).

Algorithm 10.27 (Parallel proximal algorithm (PPXA))

Fix $\varepsilon\in\,]0,1[$, $\gamma>0$, $(\omega_i)_{1\le i\le m}\in\,]0,1]^m$ such that
$\sum_{i=1}^m \omega_i=1$, $y_{1,0}\in\mathbb{R}^N,\ldots,y_{m,0}\in\mathbb{R}^N$
Set $x_0=\sum_{i=1}^m \omega_i y_{i,0}$
For $n=0,1,\ldots$

$$\left\lfloor\begin{array}{l} \text{For } i=1,\ldots,m \\ \quad\left\lfloor p_{i,n}=\operatorname{prox}_{\gamma f_i/\omega_i} y_{i,n}\right. \\ p_n=\displaystyle\sum_{i=1}^m \omega_i p_{i,n} \\ \varepsilon\le\lambda_n\le 2-\varepsilon \\ \text{For } i=1,\ldots,m \\ \quad\left\lfloor y_{i,n+1}=y_{i,n}+\lambda_n\big(2p_n-x_n-p_{i,n}\big)\right. \\ x_{n+1}=x_n+\lambda_n(p_n-x_n). \end{array}\right.$$

Proposition 10.28 [54] *Every sequence $(x_n)_{n\in\mathbb{N}}$ generated by Algorithm 10.27 converges to a solution to Problem 10.26.*

Example 10.29 (image recovery) In many imaging problems, we record an observation $y\in\mathbb{R}^M$ of an image $\overline{z}\in\mathbb{R}^K$ degraded by a matrix $L\in\mathbb{R}^{M\times K}$ and corrupted by noise. In the spirit of a number of recent investigations (see [37] and the references therein), a tight frame representation of the images under consideration can be used. This representation is defined through a synthesis matrix $F^\top\in\mathbb{R}^{K\times N}$ (with $K\le N$) such that $F^\top F=\nu I$, for some $\nu\in\,]0,+\infty[$. Thus, the original image can be written

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