Solution:

$$(a^4b^{-2})^{-2}\cdot\frac{a^6b}{a^{-2}b^5} = (a^4)^{-2}\cdot(b^{-2})^{-2}\cdot a^{6-(-2)}b^{1-5} = a^{-8}\cdot b^4\cdot a^8b^{-4} = a^{-8+8}\cdot b^{4+(-4)} = a^0b^0 = 1.$$

Exercise 1.3. Simplify the following expression containing exponentiations:

a) $2^6\cdot 2^7$ b) $\dfrac{2^6}{2^4}$ c) $10^6\cdot\left(\dfrac{1}{5}\right)^6$

d) $(2^4)^3$ e) $\dfrac{8^4}{4^4}$ f) $\dfrac{a^7\cdot a^{-4}}{a^3}$

g) $3^6\cdot 3^3 : 3^7$ h) 3^{2^5} i) $(3^2)^5$

j) $\dfrac{(2^5\cdot 3^4\cdot 5^3)\cdot(2^4\cdot 3^3\cdot 5^2\cdot 7^3)}{2^6\cdot 3^5\cdot 5^5\cdot 7}$ k) $2^{4^2} : (2^4)^3$ l) $(-3a^2b^4c^3)^2$

m) $4a^2b^3c^7\cdot(-3ab^2c^5)$ n) $\dfrac{-27a^8b^9c^7}{-3a^4b^8c^7}$ o) $\dfrac{2^{-3}a^4b^{-5}c^{-2}}{3^{-2}a^{-2}b^{-3}c^{-1}}$

p) $\dfrac{2}{3}(a^3)^2b^{-3}c^4\left[-\dfrac{1}{2}a^4b^7c^{-3}\right]$ q) $\dfrac{5a^0b^{-3}(c^{-2})^0}{2^{-1}a^{-3}b^{-5}d^3}$ r) $\dfrac{a^3b^{-5}c^3}{a^{-2}b^{-7}c}$

s) $(-2x^2y^{-3}z^{-1})^{-3}\cdot(-1x^{-1}y^4z^5)^2$ t) $3x^2(y^3)^2z^4\cdot 2(x^3y^2z)^3$ u) $\dfrac{a^{4^2}(b^3)^2}{a^{2^3}b^{2^2}}$

1.9.2 Radicals

Definition 1.19. Let a, b be real numbers, n a natural number. Suppose that if n is even, then a and b are positive. Then the **nth root of** a is denoted by $\sqrt[n]{a}$ and is defined by

$$\sqrt[n]{a} = b \quad \textbf{if and only if} \quad a = b^n.$$

For $\sqrt[2]{a}$ we use the notation $\sqrt{a}$.

Example.

$\sqrt{4} = 2,\quad \sqrt[3]{125} = 5,\quad \sqrt[5]{-32} = -2,\quad \sqrt[4]{-16}$ has no sense (it is not a real number)

Theorem 1.20. (Properties of radicals) *If n, k are positive integers and a, b are positive real numbers, then we have*

Exercise 2.1. Let $P(x,y), Q(x,y)$ and $H(x,y)$ be polynomials defined by

$$\begin{aligned} P(x,y) &:= x^2 - 3xy + y^2, \\ Q(x,y) &:= 2x^3 - x^2y + 3xy^2 + 5y^3, \\ H(x,y) &:= x^2 + 5xy + 3x - 2xy^3 + y. \end{aligned}$$

Compute the following expressions:

a) $P(x,y) + H(x,y)$

b) $xP(x,y) + Q(x,y)$

c) $P(x,y) \cdot Q(x,y)$

d) $P(x,y) - 2H(x,y)$

e) $P(x,y) \cdot H(x,y)$

f) $(P(x,y) - H(x,y))P(x,y)$

Exercise 2.2. Let $P(x), Q(x)$ and $H(x)$ be univariate polynomials defined by

$$\begin{aligned} P(x) &:= x^2 - 3x + 2, \\ Q(x,y) &:= x^3 - 2x^2 + 3x + 5, \\ H(x,y) &:= x^2 + 2. \end{aligned}$$

Compute the following expressions:

a) $P(x) + Q(x)$

b) $xP(x) + Q(x)$

c) $P(x) \cdot Q(x)$

d) $P(x) + 2H(x)$

e) $P(x) \cdot H(x)$

f) $(P(x) + Q(x)) \cdot H(x)$

g) $(x \cdot P(x) + 3 \cdot Q(x)) \cdot H(x)$

h) $P(x) + Q(x) + H(x)$

i) $P(x) \cdot (Q(x) + H(x))$

j) $P(x) \cdot Q(x) \cdot H(x)$

j) $f(x) := x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 2, \quad g(x) := x^4 - 3x^3 + x^2 - 4x + 3$

k) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 1$

l) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 1$

m) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 2$

n) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 2$

o) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 3$

p) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x^2 - 1$

q) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 1$

r) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 1$

s) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 2$

t) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 2$

u) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 3$

v) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 3$

w) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - x - 1$

x) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - 1$

y) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + x - 1$

z) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + 2x - 1$

2.2.4 Horner's scheme

In this section we consider that case of the polynomial division, when the divisor takes the form $x - c$. Let us consider a concrete example:

using Horner's scheme:

a) $f(x) = x^5 - 2x^4 + x^3 - 3x^2 + 2x - 5, \qquad g(x) = x + 1$

b) $f(x) = x^5 - 5x^4 + 3x^3 - 2x^2 + 2x - 3, \qquad g(x) = x - 2$

c) $f(x) = x^5 - 5x^4 + 3x^3 - 2x^2 + 2x - 3, \qquad g(x) = x + 2$

d) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x - 1$

e) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x + 1$

f) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x - 2$

g) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x + 2$

h) $f(x) = x^6 - 2, \qquad g(x) = x - 1$

i) $f(x) = x^6 - 2, \qquad g(x) = x - 2$

j) $f(x) = x^6 - 2, \qquad g(x) = x + 2$

k) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x - 1$

l) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 1$

m) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x - 2$

n) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 2$

o) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 3$

p) $f(x) := x^7 - x^5 + x^3 - x, \qquad g(x) := x + 1$

q) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 1$

r) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 1$

s) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 2$

t) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 2$

u) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 3$

v) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 3,$

w) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x + 1$

x) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x - 1$

y) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x + 2$

z) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 6, \qquad g(x) = x + 1$

Exercise 2.6. Decide using Horner's scheme if the below polynomial $f(x)$ is divisible by the polynomial $g(x)$ or not, and if the answer is yes, then compute the quotient $\frac{f(x)}{g(x)}$, and if the answer is no, then compute the quotient and the remainder ficients:

a) $3x + 18x^3y^3 + 27x^5y^6$

b) $x^3y^2 - 100x - x^2y^3 + 100y + x^2y^2z - 100z$

c) $9a^4 + 41a^2 - 20$

d) $(a^2b^2 + 1)^2 - (a^2 + b^2)^2$

e) $(x + 2y)^3 + (3x - y)^3$

f) $x^8 + x^4 + 1$

g) $(x + y)^4 + x^4 + y^4$

h) $x^4 - 2(a^2 + b^2)x^2 + a^4 + b^4 - 2a^2b^2$

i) $x^2 + 3x - x^4 - 3x$

j) $x^5 - 5x^4 + 4x^3 - x^2 + 5x - 4$

k) $x^2 + 2xy + y^2 - xz - yz$

l) $(abc + abd + acd + bcd)^2 - abcd(a + b + c + d)^2$

m) $3x^4y^4 - x^8 - y^8$

n) $(ac + pbd)^2 + p(ad - bc)^2$

o) $ac^2 - ab^2 + b^2c - c^3$

p) $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2$

q) $a^2b^4c^2 - a^2b^2c^4 + a^4b^2c^2 - a^4b^4$

r) $a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2 - 4abcd$

s) $x^5 + 2x^4 + 3x^2 + 2x + 1$

t) $9x^6 + 18x^5 + 26x^4 + 16x^3 + 6x^2 - 2x - 1$

u) $(x + y)^3 + 3(x + y)(x^2 - y^2) + 3(x - y)(x^2 - y^2) + (x - y)^3 - 27y^3$

v) $(cx + by)(ax + cy)(bx + ay) - (bx + cy)(cx + ay)(ax + by)$

w) $(x^2 + x + 1)(x^2 + x + 2) - 12$

x) $abc(a + b + c) - ab - ac - bc - a^2b^2c^2 + 1$

y) $(x^2 + x + 1)(x^3 + x^2 + 1) - 1$

z) $(x - a)^3(b - c) + (x - b)^3(c - a) + (x - c)^3(a - b) + 3x(b - c)(c - a)(a - b)$

2.3.5 Divisibility of polynomials

Definition 2.4. Let $\mathbb{T}$ be any of the sets $\mathbb{Q}, \mathbb{R}$. Let $P(x), Q(x) \in \mathbb{T}[x]$ be two polynomials. We say that Q **divides** P if there exists a polynomial $R(x) \in \mathbb{T}[x]$ such that $P(x) = Q(x)R(x)$. Further, if Q divides P we also say that P **is divisible by** Q, or Q **is a factor of** P.

Notation. For Q **divides** P we use the notation $Q \mid P$ or $Q(x) \mid P(x)$.

Exercise 2.10. Simplify the following expressions

a) $\dfrac{30x^2y}{6xy^2}$ b) $\dfrac{5x^2y^7z^3}{10x^4yw^2}$

c) $\dfrac{-12x^4}{24x^6}$ d) $\dfrac{3(-x)^5}{12(-x)^6}$

e) $\dfrac{x-3}{2(x-3)^2}$ f) $\dfrac{5(x-1)^2}{x^2-1}$

g) $\dfrac{9x^2+3xy}{3xy+9y^2}$ h) $\dfrac{x^2-8x}{x^3-8x^2}$

i) $\dfrac{5x-20}{x^2-16}$ j) $\dfrac{7x^4-7y^4}{9x^2y^2+9y^4}$

> We may multiply both the numerator and the denominator of an algebraic fraction by a non-zero expression, and we get a fraction which is equivalent to the original fraction.

Example. Here we amplify the fraction $\frac{x+1}{x-1}$ by x^2+x+1:

$$\frac{x+1}{x-1} = \frac{(x+1)(x^2+x+1)}{(x-1)(x^2+x+1)} = \frac{(x+1)(x^2+x+1)}{x^3-1}.$$

Exercise 2.11. Amplify the following expressions so that they have the same denominator

a) $\dfrac{3}{5a^2b^7}$ and $\dfrac{1}{a^3b}$

b) $\dfrac{1}{a^3(x+1)}$ and $\dfrac{1}{a^2(x+1)^2}$

c) $\dfrac{x+3}{2x-1}$ and $\dfrac{x-1}{3x+2}$

d) $\dfrac{1}{x+1}$, $\dfrac{1}{a^3}$ and $\dfrac{1}{3a}$

2.4.3 Addition and subtraction of rational algebraic expressions

> 1. If we have to **add (subtract)** two or more rational algebraic expressions, **whose denominators are the same**, then the result is a fraction whose denominator is the same like the common denominator of the summands, and the numerator is the sum (difference) of the original numerators.
>
> 2. If we have to **add (subtract)** two or more rational algebraic expressions, **whose denominators are different** then we first amplify the fractions so that all denominators become the same expression, and we use the rule described in 1.
>
> 3. When choosing the common denominator, we have to try to find the most simple such expression, i.e. the least common multiple of all the denominators.

The above rules can be summarized by the above formulas. If the denominators are the same, then

$$\frac{a}{d} + \frac{c}{d} = \frac{a+c}{d}, \qquad \frac{a}{d} - \frac{c}{d} = \frac{a-c}{d}.$$

In the case when the denominators are different, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd},$$
$$\frac{a}{be} + \frac{c}{de} = \frac{ad}{bde} + \frac{bc}{bde} = \frac{ad+bc}{bde}.$$

Remark. The product of the denominators is always a theoretically possible choice for the common denominator, however we strongly discourage the student to choose this way, since it may make the solution much more complicated. It is

i) $\left[\left(\frac{2a}{b}-\frac{b}{2a}\right)^2+2\right]\frac{2ab}{16a^4+b^4}$

j) $\left(\frac{5a}{a+x}+\frac{5x}{a-x}+\frac{10ax}{a^2-x^2}\right):\left(\frac{a}{a+x}+\frac{x}{a-x}+\frac{2ax}{a^2-x^2}\right)$

k) $\left(\frac{b^2}{a^3-ab^2}+\frac{1}{a+b}\right):\left(\frac{a-b}{a^2+ab}-\frac{a}{b^2+ab}\right)$

l) $\left(x-\frac{4xy}{x+y}+y\right):\left(\frac{x}{x+y}-\frac{y}{y-x}-\frac{2xy}{x^2-y^2}\right)$

m) $\left(\frac{(a+b)^2+2b^2}{a^3-b^3}-\frac{1}{a-b}+\frac{a+b}{a^2+ab+b^2}\right)\cdot\left(\frac{1}{b}-\frac{1}{a}\right)$

n) $\frac{x^2-1}{xy}:\left[\left(\frac{x^2-xy}{x^2y+y^3}-\frac{2x^2}{y^3-xy^2+x^2y-x^3}\right)\cdot\left(1-\frac{y-1}{x-\frac{y}{x^2}}\right)\right]$

o) $\dfrac{x^3-\left(\frac{1}{1+\frac{1}{x}}+\frac{1}{\frac{1}{x}+\frac{1}{x^2}}\right):\left(\frac{1}{\frac{1}{x}-1}-\frac{1}{\frac{1}{x^2}-\frac{1}{x}}\right)}{x^3-1}$

p) $\dfrac{\left(\frac{x+y^2}{a+b}-\frac{x^4-y^4}{a^3+b^3}:\frac{x^2-y^2}{a^2-ab+b^2}\right)\cdot(2a+2b-ax-bx)}{x^2-3x+3}$

q) $\dfrac{\frac{a^2+ab+b^2}{a^3-b^3}-\frac{a^2+2ab+b^2}{a^3+b^3}:\frac{a^2-b^2}{a^2-ab+b^2}}{(7a^3+ab+b^5)(2a-3b)-b^3+a^3}$

2.5 Algebraic expressions containing roots

When working with expressions containing roots the main difference to the case of rational expressions is that it is harder to find a suitable but simple common denominator. Thus in many cases it is useful to **rationalize the denominator** of such fractions (i.e. to get rid of the roots appearing in the denominator using equivalent transformations of the expression). This is done generally by using formulas for special products.

Here we present the most frequently used methods for rationalizing the denominator of a fraction:

Indeed, if we wish to rationalize the denominator of $\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}}$ $(a, b > 0)$ then we amplify the fraction by $(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2$ as follows:

$$\begin{aligned}\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}} &= \frac{(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left((\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2\right)} \\ &= \frac{(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{a-b}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}} &= \frac{(\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left((\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2\right)} \\ &= \frac{(\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{a+b}\end{aligned}$$

Example. Rationalize the denominator of the following fractions

1). $\frac{3}{2\sqrt{5}}$

$$\frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}} = \frac{3\sqrt{5}}{10}$$

2). $\frac{1}{2\sqrt[5]{3}}$

$$\frac{1}{2\sqrt[5]{3}} = \frac{\left(\sqrt[5]{3}\right)^4}{2\sqrt[5]{3}\cdot\left(\sqrt[5]{3}\right)^4} = \frac{\sqrt[5]{81}}{6}$$

3). $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

$$\begin{aligned}\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}} &= \frac{\sqrt{6}(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} \\ &= \frac{\sqrt{6}(\sqrt{2}-\sqrt{3})}{2-3} = \sqrt{6}(\sqrt{3}-\sqrt{2}) = 3\sqrt{2}-2\sqrt{3}\end{aligned}$$

- Solve the following linear equation over the real numbers

$$\frac{x-2}{3}+\frac{x}{4}=\frac{7x-8}{12}.$$

Solution. We follow the general strategy described above:

$$\begin{aligned}
&\frac{x-2}{3}+\frac{x}{4}=\frac{7x-8}{12} && /\cdot 12\\
&4(x-2)+3x=7x-8\\
&4x-8+3x=7x-8\\
&7x-8=7x-8 && /\ +8-7x\\
&0=0 \quad \textbf{identity}
\end{aligned}$$

We have used equivalent transformations, and our equation reduces to an identity, which means that every real number for which the original equation is defined (i.e. the expressions in the equation have sense) is a solution to this equation. In our case every expression in the original equation has sense for every real number.

This means the solutionset is the whole set of real numbers: $S=\mathbb{R}$.

- Solve the following linear equation over the real numbers

$$\frac{x-2}{3x}+\frac{1}{4}=\frac{7x-8}{12x}.$$

Solution. First we have to put conditions:

$$3x\neq 0 \quad \text{and} \quad 12x\neq 0$$

which leads to $x\in\mathbb{R}\setminus\{0\}$.

which means that $x = -5$. So we have to write -5 in the first row of our table, and then below the -5 we put 0 in the second row. We fill by $+$ (the sign opposite to the sign of -5) the left half of the second row, and by $-$ (the sign of -5) the right half of the second row.

	$-\infty$	-5	∞
$-2x-10$	+ + + + + + + + + + + +	0	- - - - - - - - - - - - - - - - - - - -

Now it is easy to read the result from the above table.

4.3.2 The sign of quadratic expressions

In this section we consider the expression ax^2+bx+c with $a, b, c \in \mathbb{R}$ and $a \neq 0$, and we prove a theorem which summarizes all possible cases for determining the sign of this expression, depending on the values of x. We have to split the discussion in three cases depending on the sign of the discriminant.

Theorem 4.8. *Let $\Delta = b^2 - 4ac$ be the discriminant of the polynomial ax^2+bx+c. The below three simple tables of signs describe the sign of the expression ax^2+bx+c where $a, b, c \in \mathbb{R}$ and $a \neq 0$.*

1). If the discriminant is negative, i.e. $\Delta < 0$, then the equation $ax^2+bx+c = 0$ has no real zeros, and we have

	$-\infty$		∞
ax^2+bx+c	*sign of a*	*sign of a*	*sign of a*

2). If the discriminant is zero, i.e. $\Delta = 0$, then the equation $ax^2+bx+c = 0$ has two coinciding zeros, $x_1 = x_2$, and we have

Example. Using a table of signs describe the sign of the expression $-2x^2+6x-4$.

Solution. Clearly $-2x^2+6x-4$ takes zero if and only if

$$-2x^2+6x-4=0$$

which means that $x_1=1$ or $x_2=2$. So we have to write 1 and 2 in the first row of our table, and then below them we put 0 in the second row. We fill by $+$ (the sign opposite to the sign of the leading coefficient $a=-2$) the the second row between the two zeros, and by $-$ (the sign of $a=-2$) the two sides of the second row.

	$-\infty$		1		2		∞
$-2x^2+6x-4$		- - - - - - - - - - - - - - - -	0	+ + + + + + +	0	- - - - - - - -	

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $2x^2+8x+8$.

Solution. Clearly $2x^2+8x+8$ takes zero if and only if

$$2x^2+8x+8=0$$

which means that $x_1=x_2=-2$. So we have to write -2 in the first row of our table, and then below the -2 we put 0 in the second row. Finally, we fill by $+$ (the sign of the leading coefficient $a=2$) the the second row around the 0.

	$-\infty$		-2		∞
$2x^2+8x+8$		+ + + + + + + + + + + +	0	+ + + + + + + + + + + +	

Now it is easy to read the result from the above table.

$$\begin{aligned}
&-(x+1)-(x-2)+(3-x)=6\\
&-x-1-x+2+3-x=6\\
&-3x=2\\
&x=-\frac{2}{3}\notin I_1
\end{aligned}$$

Since the solution we got is not in the interval defining Case 1, thus the solution set of the first case is

$$S_1=\emptyset.$$

Case 2. If $x\in[-1,2[\,:=I_2$ then

$$\begin{aligned}
&(x+1)-(x-2)+(3-x)=6\\
&x+1-x+2+3-x=6\\
&-x=0\\
&x=0\in I_2
\end{aligned}$$

So the solution set of the second case is

$$S_2=\{0\}.$$

Case 3. If $x\in[2,3[\,:=I_3$ then

$$\begin{aligned}
&(x+1)+(x-2)+(3-x)=6\\
&x=4\notin I_3
\end{aligned}$$

So the solution set of the third case is

$$S_3=\emptyset.$$

Case 4. If $x \in [3, \infty[:= I_4$ then

$$
\begin{aligned}
&(x+1)+(x-2)-(3-x)=6\\
&x+1+x-2-3+x=6\\
&3x=10\\
&x=\frac{10}{3}\in I_4
\end{aligned}
$$

So the solution set of the third case is

$$S_4 = \left\{\frac{10}{3}\right\}.$$

To get the complete solution set of the equation we have to join the sets of solutions of the above subcases, i.e.

$$S = S_1 \cup S_2 \cup S_3 \cup S_4 = \left\{0, \frac{10}{3}\right\}.$$

For the graph of $|x+1|+|x-2|+|3-x|-6$ see Figure 5.1.

Example. Solve the following equation in $x \in \mathbb{R}$

$$|x-1|+|x+1|=2.$$

Solution. We shall do equivalent transformations. We have to get rid of the absolute values by the formulas

$$|x-1| = \begin{cases} (x-1) & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 \leq 0. \end{cases} \qquad |x+1| = \begin{cases} (x+1) & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 \leq 0. \end{cases}$$

We build a table of signs containing all expressions appearing in absolute value:

$$
\begin{aligned}
&x-1=0 \qquad\qquad &x+1=0\\
&x_1=1 \qquad\qquad &x_2=-1
\end{aligned}
$$

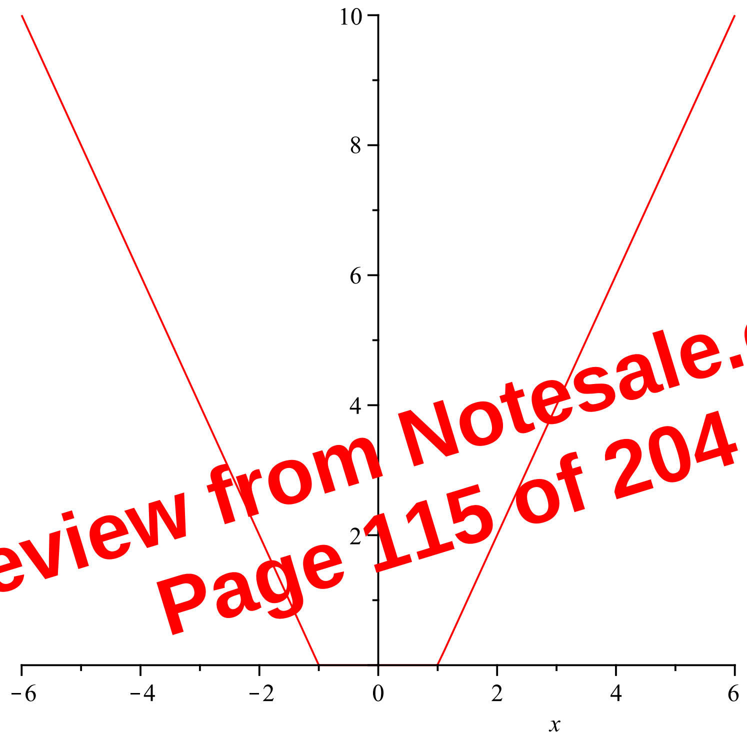

Figure 5.2: Graph of the function $|x-1|+|x+1|-2$.

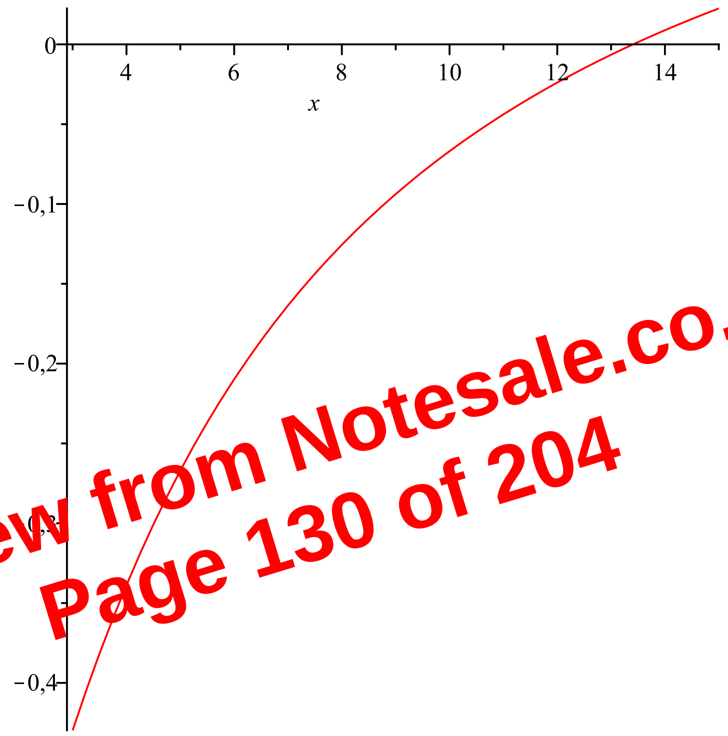

Figure 5.7: The graph of $\frac{1+x^4}{(1+x)^4} - \frac{3}{4}$ with $x \in [3, 15]$

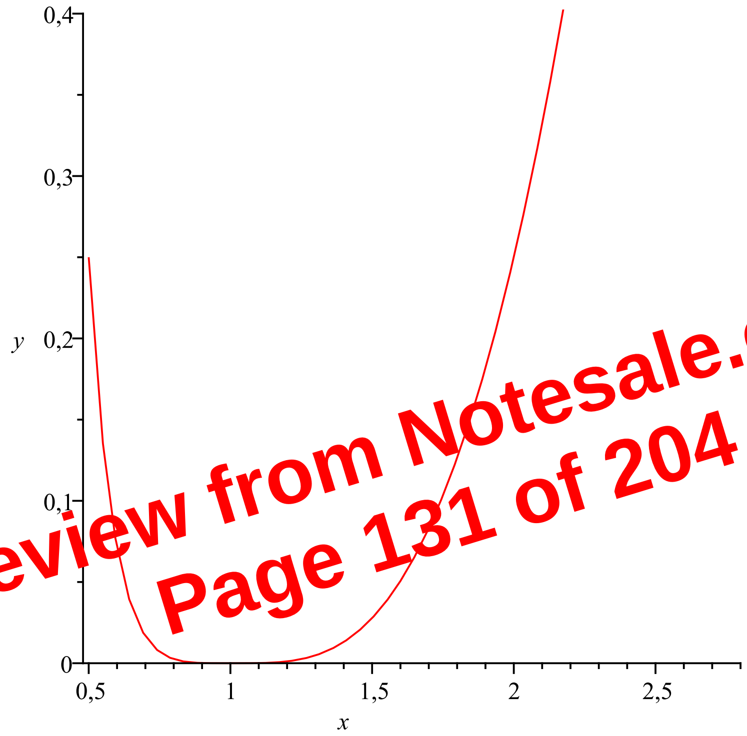

Figure 5.8: The graph of $x^2 + 6 + \frac{1}{x^2} - 4x - \frac{4}{x}$

that is

$$y^2 + y - 12 = 0$$

which gives $y_1 = -4, y_2 = 3$. The negative value is impossible, the positive value gives

$$x^2 + x - 2 = 0$$

and we get

$$x_1 = 1, x_2 = -2.$$

Now, we need to check these candidates for the solution. If $x = 1$, then we have

$$1^2 + 1 + \sqrt{1^2 + 1 + 7} = 5,$$

this is a solution of our initial equation. If $x = -2$, then we obtain

$$(-2)^2 - 2 + \sqrt{(-2)^2 - 2 + 7} = 5,$$

so $x = -2$ is also a root of the original equation.

5.4 Exponential equations

Definition 5.7. An equation is called *exponential equation* if the unknown (or unknowns) appears in the exponents of algebraic expressions.

The following theorem is our basic tool for solving exponential equation.

Theorem 5.8. *Let b be a positive real number with $b \neq 1$. Then*

$$b^x = b^y \qquad \textit{implies} \qquad x = y$$

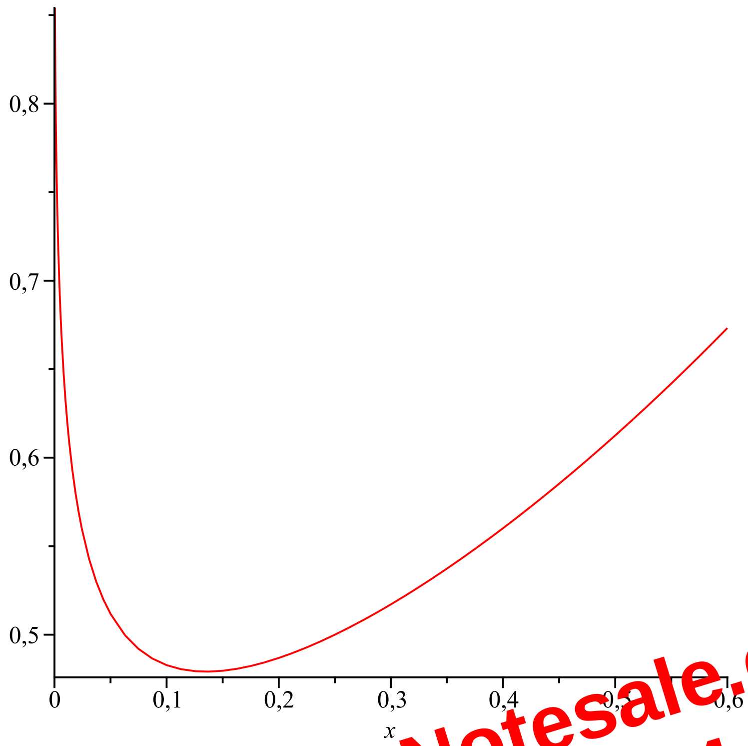

Figure 5.12: The graph of $x^{x^{1/2}}$.

Example. Solve the equation

$$x^{x^{1/2}} = 1/2.$$

Solution. We will introduce a new variable, let $y = x^2$. Rewriting our equation we have

$$y^{2y} = \frac{1}{2}$$

One can prove that the function $f(y) = y^{2y}$ is strictly decreasing on the interval $0 < y \leq \frac{1}{e}$ and strictly increasing if $y \geq \frac{1}{e}$. This fact gives that we have at most two solutions and after substitution we obtain exactly two solutions for y, $\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{1}{4}$ s $\frac{1}{16}$ for x. See Figure 5.12!

we have

$$\log_2(4x^2) = 2.$$

Exponentiating both sides (with base 2), we obtain

$$2^{\log_2(4x^2)} = 2^2 = 4.$$

Using the fact that

$$a^{\log_a x} = x,$$

we can simplify the left-hand side to get

$$4x^2 = 4.$$

Dividing both sides by 4, we obtain

$$x^2 = 1.$$

Solving for x, we see that $x = \pm 1$. We check both of our candidate solutions. If $x = 1$, we have

$$\log_2(4 \cdot 1) - \log_2 1 = \log_2 4 - 0 = 2,$$

so $x = 1$ is a solution. If $x = -1$, we have

$$\log_2(4 \cdot (-1)) - \log_2(-1) = \log_2(-4) - \log_2(-1).$$

However, neither expression is defined, since the domain of the logarithm function does not contain negative numbers. Thus, the only solution to the above equation is $x = 1$.

Example. Solve the equation

$$\lg(x + 2) + \lg(x - 1) = 1$$

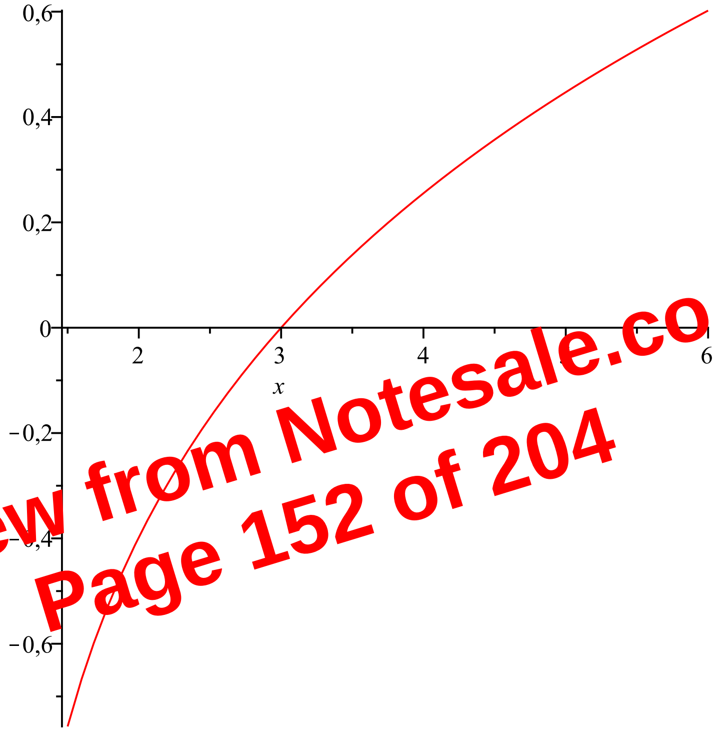

Figure 5.13: The graph of $\lg(x+2)+\lg(x-1)-1$

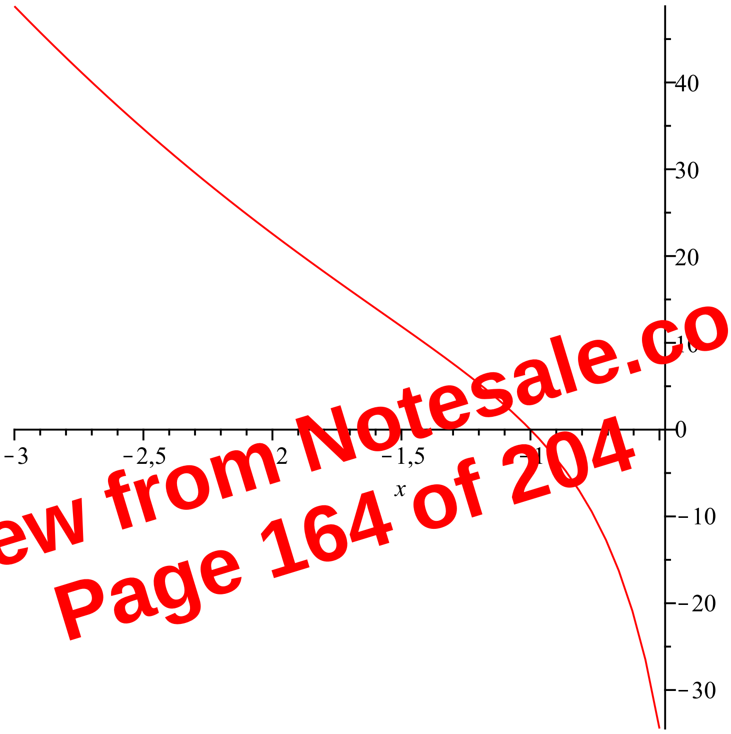

Figure 6.3: The graph of $5x^2 + 4 - \left(\frac{x^2-16}{5x}\right)^2$ with $-3 \leq x \leq -0.5$.

Exercise 6.1. Solve the following systems of equations in the set of real numbers:

a) $x + y = 7, xy = -18$

b) $3x + 4y = -18, xy = 6$

c) $x^2 + y^2 = 25, xy = 12$

d) $x^2 + y^2 = 25, x + y = 5$

e) $x^2 - y^2 = 20, x + y = 10$

f) $x^2 - y^2 = 40, xy = 21$

g) $x^2 + y^2 = 20, xy = 8$

h) $y - x^2 = 3, y - x = 3$

i) $4x^2 + 4y^2 = 17xy, x + y = 10$

j) $\dfrac{1}{x} + \dfrac{1}{y} = 1, x + y = 4$

k) $\dfrac{4}{x} + \dfrac{6}{y} = 0, \dfrac{3}{x} - \dfrac{4}{y} = \dfrac{17}{6}$

l) $x - xy + y = 1, xy = 20$

m) $x + y + xy = 49, (x + y)xy = 468$

n) $\log_7 x + 2\log_{\frac{1}{7}} x = \log_{49} x - 3$

o) $4y = x - 12, x + y = 2$

p) $y - 3x = 5, y + x = 3$

Chapter 8

Exercises for the interested reader

8.1 Identities of algebraic expressions

1). Let $a, b, c \in \mathbb{R}$ be real numbers. Prove that

(1) $(a+b+c)^3 - (a^3+b^3+c^3) = 3(b+c)(c+a)(a+b)$,

(2) $a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$.

2). Let $a, b, c \in \mathbb{R}$ be pairwise distinct non-zero real numbers. Prove that

(1) $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)} = 0$

(2) $\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)} = 1$

(3) $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = a+b+c$

(4) $\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} = a^2+b^2+c^2+ab+bc+ac$

(5) $\frac{a^{-1}}{(a-b)(a-c)} + \frac{b^{-1}}{(b-c)(b-a)} + \frac{c^{-1}}{(c-a)(c-b)} = \frac{1}{abc}$

(6) $\frac{a^{-2}}{(a-b)(a-c)} + \frac{b^{-2}}{(b-c)(b-a)} + \frac{c^{-2}}{(c-a)(c-b)} = \frac{ab+bc+ac}{a^2b^2c^2}$

3). Let $a, b, c, d \in \mathbb{R}$ be pairwise distinct non-zero real numbers. Prove that

6). Compute the value of the expression

$$\left(\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}\right)\cdot\left(\frac{c}{a-b}+\frac{a}{b-c}+\frac{b}{c-a}\right)$$

provided that

(1) $a+b+c=0$,

(2) $|c|=|a-b|$.

7). Let $a,b,c\in\mathbb{R}$ be real numbers. Prove that if $(a+b+c)^3=(a^3+b^3+c^3)$ then for every $n\in\mathbb{N}$ we have

$$(a+b+c)^{2n+1}=a^{2n+1}+b^{2n+1}+c^{2n+1}.$$

8). Let $a,b,c\in\mathbb{R}$ be real numbers, such that the expressions below have sense. Prove that if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$$

then for every $n\in\mathbb{N}$ we have

$$\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}.$$

9). Let $a,b,c\in\mathbb{R}$ be real numbers. Prove that if $a^3+b^3+c^3=3abc$ then either $a+b+c=0$ or $a=b=c$.

8.2 Inequalities of algebraic expressions

1). Let $a,b,c\in\mathbb{R}$ be positive real numbers. Prove that

(1) $(a+b)(a+c)(c+a)\geq 8abc$,

(2) $(a^2+b^2)c+(b^2+c^2)a+(c^2+a^2)b\geq 6abc$,

(3) $2(a^3+b^3+c^3) \geq (a+b)ab+(b+c)bc+(c+a)ca.$

2). Let $a, b, c \in \mathbb{R}$ be positive real numbers with $a+b+c=1$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9.$$

Under what conditions do we have equality above?

3). Let $a, b \in \mathbb{R}$ be positive real numbers with $a+b=2$. Prove that

$$a^4+b^4 \geq 2.$$

4). Let $a, b \in \mathbb{R}$ be positive real numbers, and $m, n \in \mathbb{N}$ be natural numbers of the same type of parity. Prove that

$$(1) \qquad \frac{a^m+b^m}{2} \cdot \frac{a^n+b^n}{2} \leq \frac{a^{m+n}+b^{m+n}}{2}$$

$$(2) \qquad \frac{a+b}{2} \cdot \frac{a^2+b^2}{2} \cdot \frac{a^3+b^3}{2} \leq \frac{a^6+b^6}{2}$$

5). Let $a, b, c \in \mathbb{R}$ be positive real numbers. Prove that

$$a+b+c \leq \frac{a^4+b^4+c^4}{abc}$$

6). Let $a, b, c, d \in \mathbb{R}$ be positive real numbers. Prove that

$$\sqrt{(a+c)(b+d)} \leq \sqrt{ab}+\sqrt{cd}.$$

7). Let $a, b, c \in \mathbb{R}$ be positive real numbers. Prove that

$$\frac{ab}{a+c}+\frac{bc}{b+c}+\frac{ca}{c+a}=\frac{a+b+c}{2}.$$

Under what conditions do we have equality above?

8). Prove that

Results of Exercise 2.4 Denoting the quotient by $q(x)$ and the remainder by

$r(x)$ the results of the divisions are:

a) $q(x) = x + 3,$ $\quad r(x) := -2x - 1$

b) $q(x) = x^3 - 2x^2 + x + 5,$ $\quad r(x) = -3$

c) $q(x) = x^3 - 2x - 4,$ $\quad r(x) = -12x + 10$

d) $q(x) = x^3 - x - 1,$ $\quad r(x) = -5x + 3$

e) $q(x) = x^3 + 2x + 1,$ $\quad r(x) = 0$

f) $q(x) = x^4 + 2x^3 - 8x - 16,$ $\quad r(x) = 0$

g) $q(x) = x^3 + x^2 - x - 7,$ $\quad r(x) = -20x + 30$

h) $q(x) = x^3 + 7x^2 + 28x + 126,$ $\quad r(x) = 574x - 250$

i $q(x) = x^3 + 2x^2 - 4,$ $\quad r(x) = -19x + 14$

j) $q(x) = x + 1,$ $\quad r(x) = -3x^3 + 5x^2 - 2x - 1$

k) $q(x) = x^5 + 4x^4 + 2x^3 - 3x^2 - x - 4,$ $\quad r(x) = 6$

l) $q(x) = x^5 + 2x^4 - 4x^3 - x^2 + 3x - 6,$ $\quad r(x) = 16$

m) $q(x) = x^5 + 5x^4 + 8x^3 + 11x^2 + 24x + 45,$ $\quad r(x) = 100$

n) $q(x) = x^5 + x^4 - 4x^3 + 3x^2 - 4x + 5,$ $\quad r(x) = 0$

o) $q(x) = x^5 - 2x^3 + x^2 - x,$ $\quad r(x) = 10$

p) $q(x) = x^4 + 3x^3 - x^2 - 2x + 1,$ $\quad r(x) = -5x + 11$

q) $q(x) = x^5 + x^4 - 3x^3 - 4x^2 - 4x - 1,$ $\quad r(x) = -3$

r) $q(x) = x^5 - x^4 - 3x^3 + 2x^2 - 2x + 5,$ $\quad r(x) = -7$

s) $q(x) = x^5 + 2x^4 - x^2 - 2x - 1,$ $\quad r(x) = -4$

t) $q(x) = x^5 - 2x^4 - x^2 + 2x - 1,$ $\quad r(x) = 0$

u) $q(x) = x^5 + 3x^4 + 5x^3 + 14x^2 + 42x + 129,$ $\quad r(x) = 385$

v) $q(x) = x^5 - 3x^4 + 5x^3 - 16x^2 + 48x - 141,$ $\quad r(x) = 421$

w) $q(x) = x^4 + x^3 - 2x^2 - 2x - 4,$ $\quad r(x) = -3x - 6$

x) $q(x) = x^4 - 3x^2 - x - 3,$ $\quad r(x) = 2x - 5$

y) $q(x) = x^4 - x^3 - 2x^2 - 2,$ $\quad r(x) = 5x - 4$

z) $q(x) = x^4 - 2x^3 + x^2 - 5x + 11,$ $\quad r(x) = -24x + 9$

9.5 Chapter 5

Results of Exercise 5.1:

a) $x_1 = -2, x_2 = -1, x_3 = 3$

b) $x_1 = -2, x_2 = 1, x_3 = 2, x_4 = 3$

c) $x_1 = -4, x_2 = -2, x_3 = 1, x_4 = 2, x_5 = 3$

d) $x_1 = 2, x_2 = 2, x_3 = -1, x_4 = -5$

e) $x_1 = -5, x_2 = -3, x_3 = -2, x_4 = -2, x_5 = 4$

f) $x_1 = -5, x_2 = -2, x_3 = 2, x_4 = 3$

g) $x_1 = -4, x_2 = -2, x_3 = 3, x_4 = 5$

h) $x_1 = -3, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = -1 - \sqrt{3}, x_6 = -1 + \sqrt{3}$

i) $x_1 = -4, x_2 = -3, x_3 = -1, x_4 = 2, x_5 = 5$

j) $x_1 = -5, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = \dfrac{-1 + \sqrt{5}}{2}, x_6 = \dfrac{-1 - \sqrt{5}}{2}$

k) $x_1 = -1, x_2 = 2, x_3 = -4, x_4 = 8$

l) $x_1 = -1, x_2 = 2, x_3 = -4, x_4 = 8, x_5 = \dfrac{-3 - \sqrt{5}}{2}, x_6 = \dfrac{-3 + \sqrt{5}}{2}$

m) $x_1 = -4, x_2 = -3, x_3 = 1, x_4 = 1, x_5 = 2, x_6 = 5$

n) $x_1 = -4, x_2 = -3, x_3 = -2, x_4 = 2, x_5 = 3$

o) $x_1 = -5, x_2 = -2, x_3 = -2, x_4 = -1, x_5 = 1, x_6 = 5$

p) $x_1 = -2, x_2 = 2, x_3 = 3, x_4 = 3$

q) $x_1 = -3, x_2 = -3, x_3 = -2, x_4 = -1, x_5 = 2, x_6 = -\sqrt{2}, x_7 = \sqrt{2}$

r) $x_1 = -2, x_2 = 1, x_3 = 3, x_4 = 9, x_5 = -\sqrt{3}, x_6 = \sqrt{3}$

s) $x_1 = -25, x_2 = -5, x_3 = -1, x_4 = 2$

t) $x_1 = -3, x_2 = -2, x_3 = -2, x_4 = 1, x_5 = 1, x_6 = 3$

u) $x_1 = -4, x_2 = -3, x_3 = 3, x_4 = 4$

v) $x_1 = -5, x_2 = -2, x_3 = 2, x_4 = 5$

w) $x_1 = -5, x_2 = 5$

x) $x_1 = -6, x_2 = 6$

y) $x_1 = -6, x_2 = 5, x_3 = 7$

z) $x_1 = -7, x_2 = 4, x_3 = 6$

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