$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- If the $f(x)$ is zero, the second order ODE said to be **homogenous**.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Or $ay'' + by' + cy = 0$

Homogenous DE : Initial value problem

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2y = 0 \qquad y(0) = 0; \quad y'(0) = \sqrt{3}$$

Sol:

The characteristic equation of the homogenous equation above are:

$$m^2 - 2m - 2 = 0$$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2}$$

$$m = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$m = 1 + \sqrt{3}, 1 - \sqrt{3}$$

Thus, we have $m_1 = 1 + \sqrt{3}, m_2 = 1 - \sqrt{3}$as the roots are real and distinct.

Hence, the solution of this ODE is

$$y = Ae^{(1+\sqrt{3})x} + Be^{(1-\sqrt{3})x}$$

take derivatives of y

$$y' = (1 + \sqrt{3})Ae^{(1+\sqrt{3})x} + (1 - \sqrt{3})Be^{(1-\sqrt{3})x}$$

Exercises

1. $\frac{d^2y}{dx^2} = 6x^2 + 2x + 1$
2. $y'' - y' - 2y = 2e^{3x}$
3. $y'' - y' - 2y = e^{-x}$
4. $y'' + 2y' + 2y = \sin x$
5. $y'' - 7y' + 12y = \sinh 3x$
6. $y'' - 2y' + 2y = 2x^2 - 1;\quad y(0) = 1$ and $y'(0) = 2$

Example:

Find the General solution for $y'' - 2y' + y = \frac{e^x}{1+x^2}$

Solution:

Homogeneous solution:

$$y'' - 2y' + y = 0$$

Characteristic eqn: $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1$$

$$y_h(x) = Ae^x + Bxe^x$$

$$\Rightarrow y_1 = e^x \; ; \quad y_2 = xe^x$$

Wronskian value:

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$w = e[e^x + xe^x] - xe^x \cdot e^x$$

$$w = e^{2x}$$