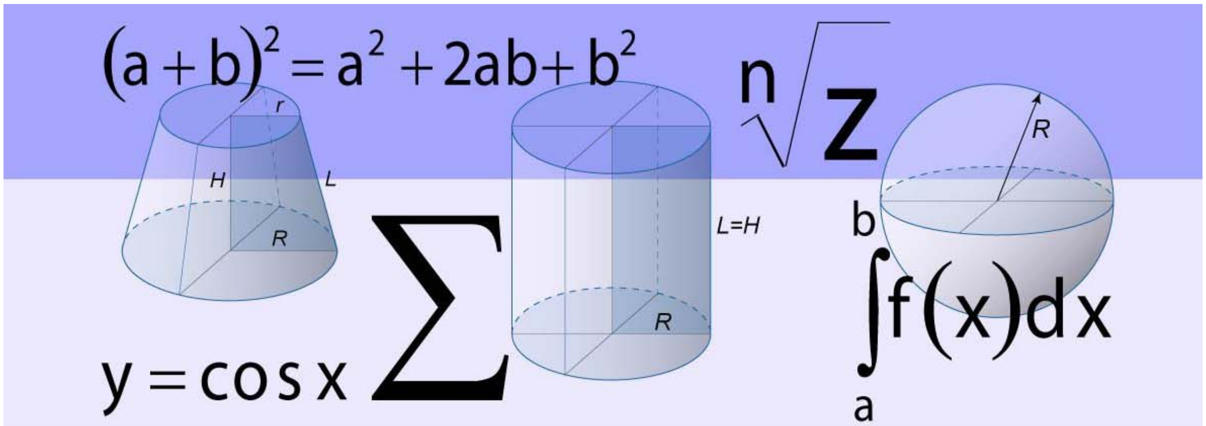


1300 Math Formulas

Alex Svirin, Ph.D.

60. Quotient in Polar Representation

$$\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} = \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right]$$

61. Power of a Complex Number

$$z^n = \left[r(\cos\varphi + i\sin\varphi)\right]^n = r^n\left[\cos(n\varphi) + i\sin(n\varphi)\right]$$

62. Formula "De Moivre"

$$(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$$

63. Nth Root of a Complex Number

$$\sqrt[n]{z} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right),$$

where

$k = 0, 1, 2, \ldots, n-1$.

64. Euler's Formula

$$e^{ix} = \cos x + i\sin x$$

2.4 Roots

Bases: a, b
Powers (rational numbers): n, m
$a, b \geq 0$ for even roots ($n = 2k$, $k \in N$)

91. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

92. $\sqrt[n]{a}\,\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$

93. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $b \neq 0$

94. $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \frac{\sqrt[nm]{a^m}}{\sqrt[nm]{b^n}} = \sqrt[nm]{\frac{a^m}{b^n}}$, $b \neq 0$.

95. $\left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$

96. $\left(\sqrt[n]{a}\right)^n = a$

97. $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$

98. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

99. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

100. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

165. $h^2 = fg$,
where h is the altitude from the right angle.

166. $m_a^2 = b^2 - \frac{a^2}{4}$, $m_b^2 = a^2 - \frac{b^2}{4}$,
where m_a and m_b are the medians to the legs a and b.

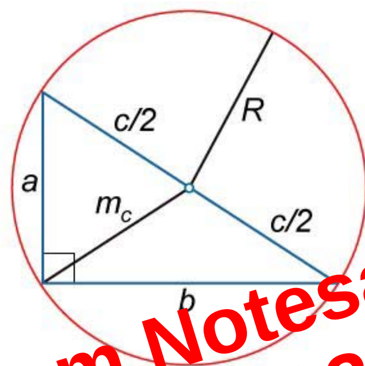


Figure 10.

167. $m_c = \frac{c}{2}$
where m_c is the median to the hypotenuse c.

168. $R = \frac{c}{2} = m_c$

169. $r = \frac{a+b-c}{2} = \frac{ab}{a+b+c}$

170. $ab = ch$

171. $S = \frac{ab}{2} = \frac{ch}{2}$

3.2 Isosceles Triangle

Base: a
Legs: b
Base angle: β
Vertex angle: α
Altitude to the base: h
Perimeter: L
Area: S

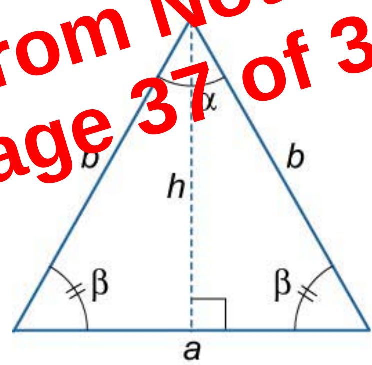


Figure 11.

172. $\beta = 90° - \frac{\alpha}{2}$

173. $h^2 = b^2 - \frac{a^2}{4}$

207. $h = b\sin\alpha = b\sin\beta$

208. $L = 2(a+b)$

209. $S = ah = ab\sin\alpha$,

$S = \frac{1}{2}d_1 d_2 \sin\varphi$.

3.8 Rhombus

Side of a rhombus: a
Diagonals: d_1, d_2
Consecutive angles: α, β
Altitude: H
Radius of inscribed circle: r
Perimeter: L
Area: S

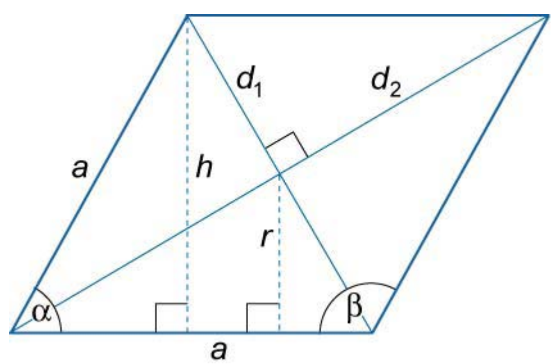


Figure 19.

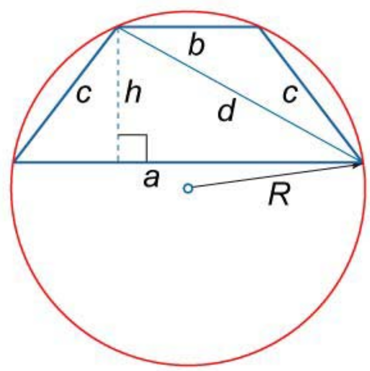


Figure 21.

218. $q = \dfrac{a+b}{2}$

219. $d = \sqrt{ab + c^2}$

220. $h = \sqrt{c^2 - \dfrac{1}{4}(b-a)^2}$

221. $R = \dfrac{c\sqrt{ab+c^2}}{\sqrt{(2c-a+b)(2c+a-b)}}$

222. $S = \dfrac{a+b}{2} \cdot h = qh$

251. $R = a$

252. $L = 6a$

253. $S = pr = \frac{a^2 3\sqrt{3}}{2}$,

where $p = \frac{L}{2}$.

3.18 Regular Polygon

Side: a
Number of sides: n
Internal angle: α
Slant height: m
Radius of inscribed circle: r
Radius of circumscribed circle: R
Perimeter: L
Semiperimeter: p
Area: S

3.21 Segment of a Circle

Radius of a circle: R
Arc length: s
Chord: a
Central angle (in radians): x
Central angle (in degrees): α
Height of the segment: h
Perimeter: L
Area: S

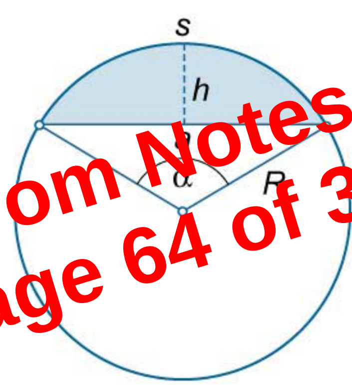


Figure 36.

271. $a = 2\sqrt{2hR - h^2}$

272. $h = R - \frac{1}{2}\sqrt{4R^2 - a^2}$, $h < R$

273. $L = s + a$

289. $S_B = \dfrac{\sqrt{3}a^2}{4}$

290. $S = \sqrt{3}a^2$

291. $V = \dfrac{1}{3}S_B h = \dfrac{a^3}{6\sqrt{2}}$.

3.26 Regular Pyramid

Side of base: a
Lateral edge: b
Height: h
Slant height: m
Number of sides: n
Semiperimeter of base: p
Radius of inscribed sphere of base: r
Area of base: S_B
Lateral surface area: S_L
Total surface area: S
Volume: V

328. $H = \sqrt{m^2 - R^2}$

329. $S_L = \pi R m = \dfrac{\pi m d}{2}$

330. $S_B = \pi R^2$

331. $S = S_L + S_B = \pi R(m + R) = \dfrac{1}{2}\pi d\left(m + \dfrac{d}{2}\right)$

332. $V = \dfrac{1}{3} S_B H = \dfrac{1}{3}\pi R^2 H$

3.33 Frustum of a Right Circular Cone

Radius of bases: R, r
Height: H
Slant height: m
Scale factor: k
Area of bases: S_1, S_2
Lateral surface area: S_L
Total surface area: S
Volume: V

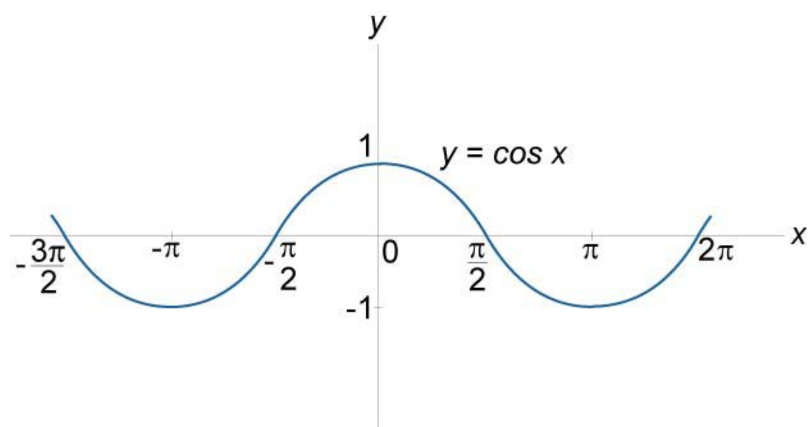


Figure 60.

375. Tangent Function

$$y = \tan x,\ x \neq (2k+1)\frac{\pi}{2},\ -\infty \leq \tan x \leq \infty.$$

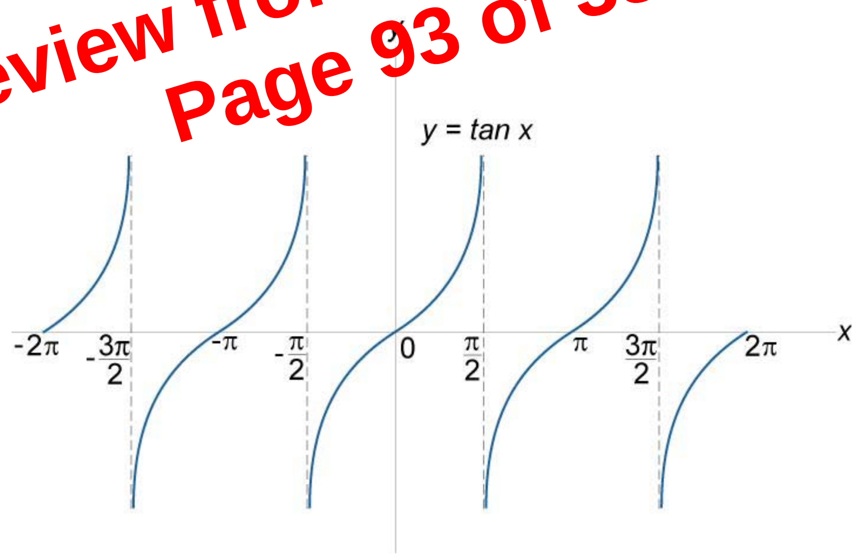


Figure 61.

4.3. Signs of Trigonometric Functions

379.

Quadrant	Sin α	Cos α	Tan α	Cot α	Sec α	Cosec α
I	+	+	+	+	+	+
II	+					+
III			+	+		
IV		+			+	

380.

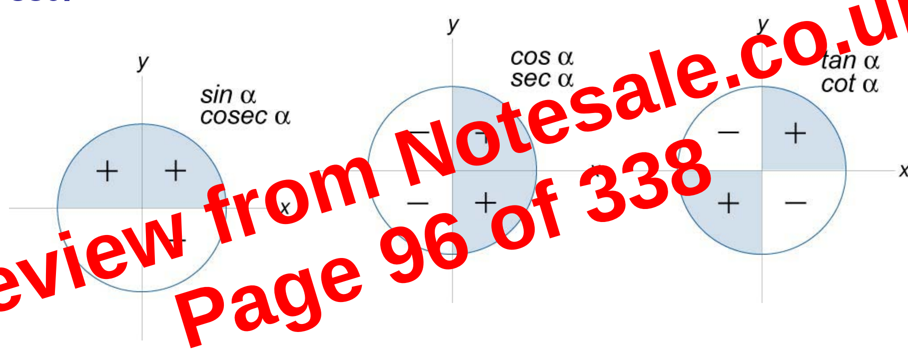


Figure 65.

425. $\cot 5\alpha = \dfrac{1 - 10\tan^2\alpha + 5\tan^4\alpha}{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}$

4.12 Half Angle Formulas

426. $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

427. $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

428. $\tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha} = \csc\alpha - \cot\alpha$

429. $\cot\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}} = \dfrac{\sin\alpha}{1-\cos\alpha} = \dfrac{1+\cos\alpha}{\sin\alpha} = \csc\alpha + \cot\alpha$

4.13 Half Angle Tangent Identities

430. $\sin\alpha = \dfrac{2\tan\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}}$

448. $1+\sin\alpha = 2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

449. $1-\sin\alpha = 2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

4.15 Transforming of Trigonometric Expressions to Sum

450. $\sin\alpha\cdot\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

451. $\cos\alpha\cdot\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

452. $\sin\alpha\cdot\cos\beta = \frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$

453. $\tan\alpha\cdot\tan\beta = \frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}$

454. $\cot\alpha\cdot\cot\beta = \frac{\cot\alpha+\cot\beta}{\tan\alpha+\tan\beta}$

455. $\tan\alpha\cdot\cot\beta = \frac{\tan\alpha+\cot\beta}{\cot\alpha+\tan\beta}$

4.16 Powers of Trigonometric Functions

456. $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$

457. $\sin^3 \alpha = \frac{3\sin \alpha - \sin 3\alpha}{4}$

458. $\sin^4 \alpha = \frac{\cos 4\alpha - 4\cos 2\alpha + 3}{8}$

459. $\sin^5 \alpha = \frac{10\sin \alpha - 5\sin 3\alpha + \sin 5\alpha}{16}$

460. $\sin^6 \alpha = \frac{10 - 15\cos 2\alpha + 6\cos 4\alpha - \cos 6\alpha}{32}$

461. $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$

462. $\cos^3 \alpha = \frac{3\cos \alpha + \cos 3\alpha}{4}$

463. $\cos^4 \alpha = \frac{\cos 4\alpha + 4\cos 2\alpha + 3}{8}$

464. $\cos^5 \alpha = \frac{10\cos \alpha + 5\sin 3\alpha + \cos 5\alpha}{16}$

465. $\cos^6 \alpha = \frac{10 + 15\cos 2\alpha + 6\cos 4\alpha + \cos 6\alpha}{32}$

469. Inverse Cotangent Function

$y = \operatorname{arccot} x, \ -\infty \le x \le \infty, \ 0 < \operatorname{arccot} x < \pi.$

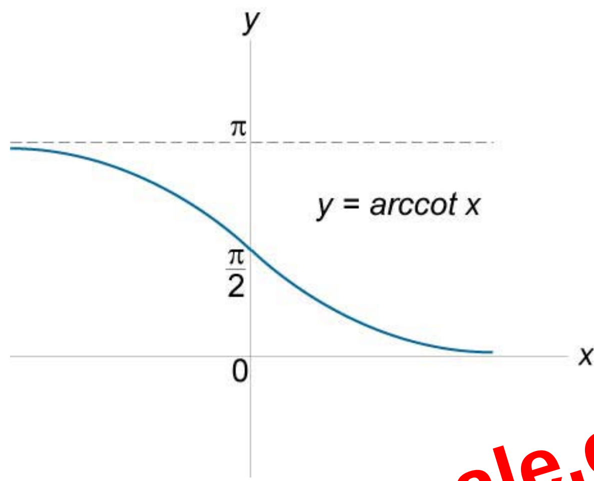


Figure 69.

470. Inverse Secant Function

$y = \operatorname{arcsec} x, \ x \in (-\infty, -1] \cup [1, \infty), \ \operatorname{arcsec} x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$

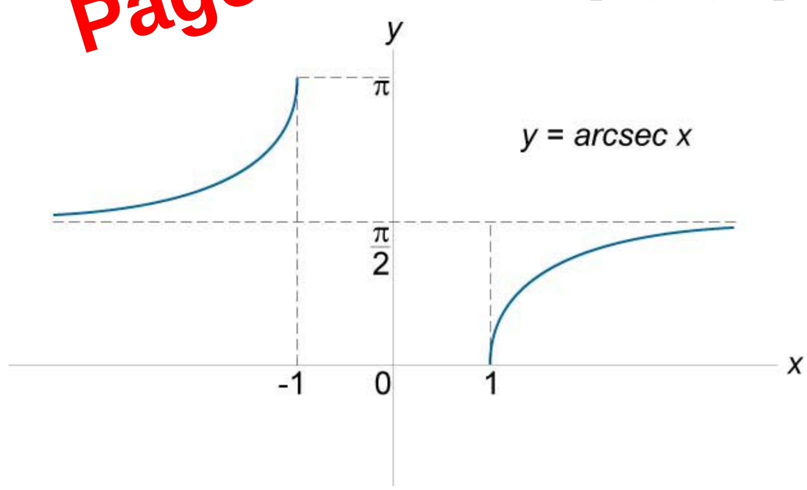


Figure 70.

471. Inverse Cosecant Function

$$y=\operatorname{arccsc} x,\ \ x\in(-\infty,-1]\cup[1,\infty),\ \operatorname{arccsc} x\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right].$$

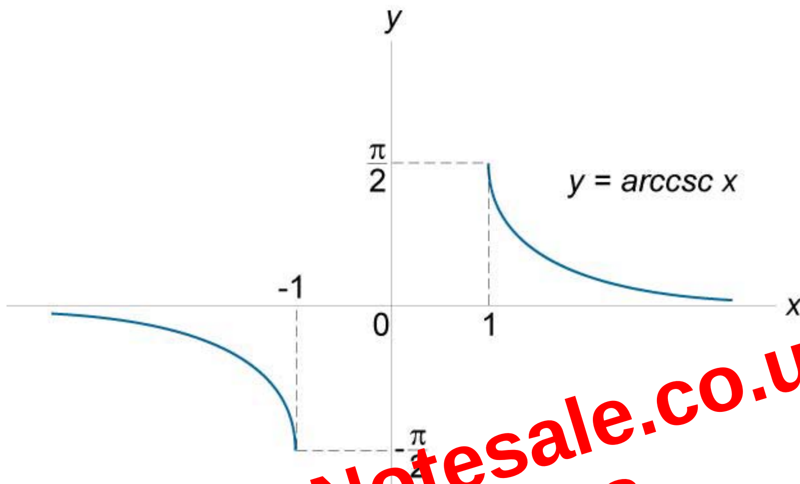


Figure 71.

4.18 Principal Values of Inverse Trigonometric Functions

472.

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
arcsin x	0°	30°	45°	60°	90°
arccos x	90°	60°	45°	30°	0°
x	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	
arcsin x	−30°	−45°	−60°	−90°	
arccos x	120°	135°	150°	180°	

493. $\arctan x = \frac{\pi}{2} - \arctan\frac{1}{x},\ x > 0.$

494. $\arctan x = -\frac{\pi}{2} - \arctan\frac{1}{x},\ x < 0.$

495. $\arctan x = \operatorname{arccot}\frac{1}{x},\ x > 0.$

496. $\arctan x = \operatorname{arccot}\frac{1}{x} - \pi,\ x < 0.$

497. $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$

498. $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$

499. $\operatorname{arccot} x = \arcsin\frac{1}{\sqrt{1+x^2}},\ x > 0.$

500. $\operatorname{arccot} x = \pi - \arcsin\frac{1}{\sqrt{1+x^2}},\ x < 0.$

501. $\operatorname{arccot} x = \arccos\frac{x}{\sqrt{1+x^2}}$

502. $\operatorname{arccot} x = \arctan\frac{1}{x},\ x > 0.$

503. $\operatorname{arccot} x = \pi + \arctan\frac{1}{x},\ x < 0.$

609. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2}, \ \lambda = 1.$$

7.2 Two-Dimensional Coordinate System

Point coordinates: x_0, x_1, x_2, y_0, y_1, y_2
Polar coordinates: r, φ
Real number: λ
Positive real numbers: a, b, c,
Distance between two points: d
Area: S

610. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

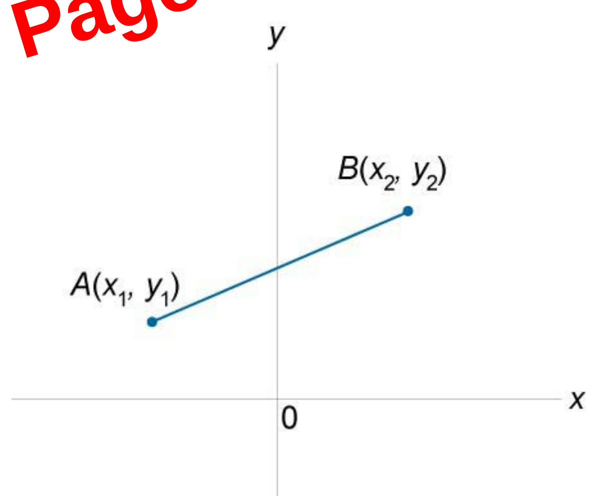


Figure 88.

7.3 Straight Line in Plane

Point coordinates: $X, Y, x, x_0, x_1, y_0, y_1, a_1, a_2, \ldots$
Real numbers: $k, a, b, p, t, A, B, C, A_1, A_2, \ldots$
Angles: α, β
Angle between two lines: φ
Normal vector: $\vec{n}$
Position vectors: $\vec{r}, \vec{a}, \vec{b}$

622. General Equation of a Straight Line
$Ax + By + C = 0$

623. Normal Vector to a Straight Line
The vector $\vec{n}(A, B)$ is normal to the line $Ax + By + C = 0$.

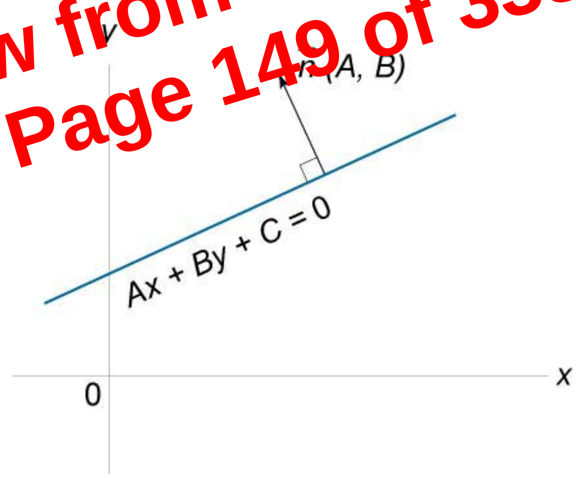


Figure 98.

624. Explicit Equation of a Straight Line (Slope-Intercept Form)
$y = kx + b$.

The gradient of the line is $k = \tan\alpha$.

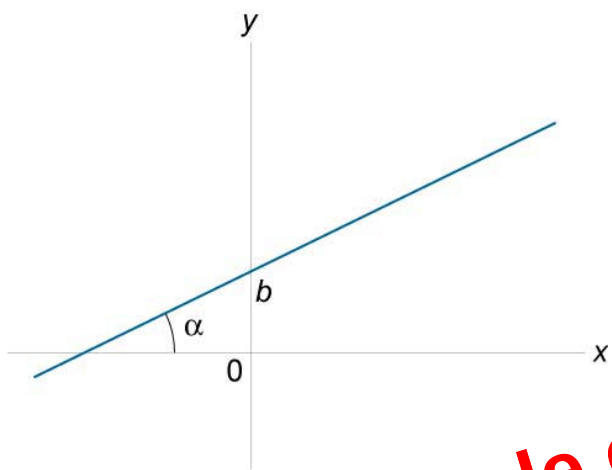


Figure 99.

625. Gradient of a Line

$$k = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

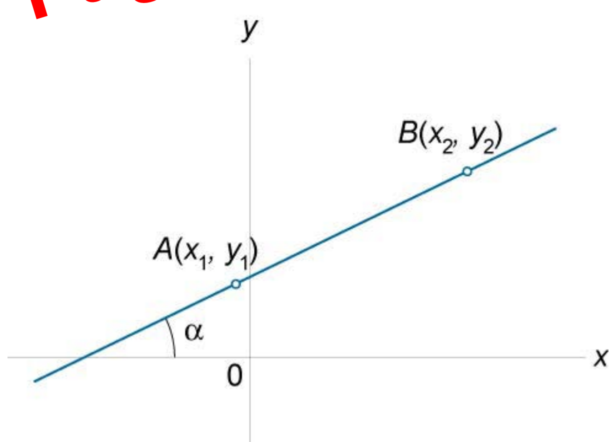


Figure 100.

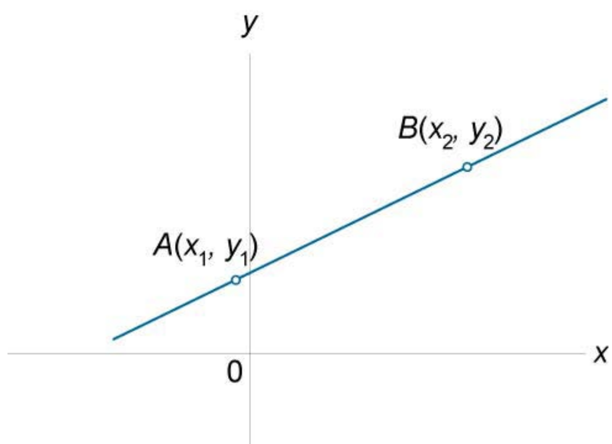


Figure 102.

628. Intercept Form

$$\frac{x}{a}+\frac{y}{b}=1$$

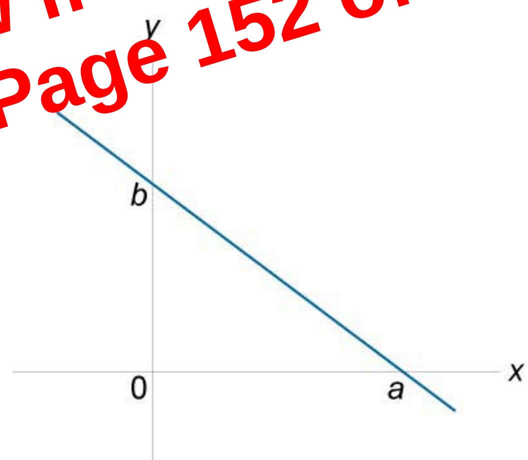


Figure 103.

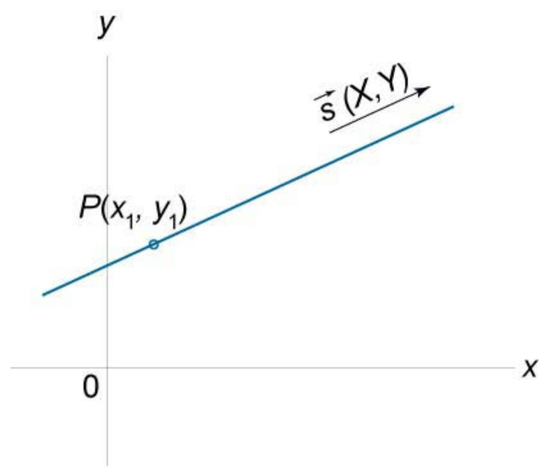


Figure 105.

631. Vertical Line
$x = a$

632. Horizontal Line
$y = b$

633. Vector Equation of a Straight Line
$\vec{r} = \vec{a} + t\vec{b}$,
where
O is the origin of the coordinates,
X is any variable point on the line,
$\vec{a}$ is the position vector of a known point A on the line ,
$\vec{b}$ is a known vector of direction, parallel to the line,
t is a parameter,
$\vec{r} = \overrightarrow{OX}$ is the position vector of any point X on the line.

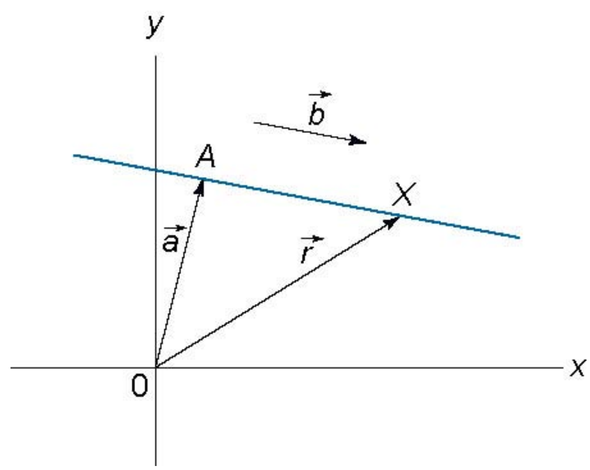


Figure 106.

634. Straight Line in Parametric Form

$$\begin{cases} x = a_1 + tb_1 \\ y = a_2 + tb_2 \end{cases},$$

where

(x, y) are the coordinates of any unknown point on the line,
(a_1, a_2) are the coordinates of a known point on the line,
(b_1, b_2) are the coordinates of a vector parallel to the line,
t is a parameter.

$$R = \sqrt{\frac{D^2 + E^2 - 4AF}{2|A|}}.$$

7.5 Ellipse

Semimajor axis: a
Semiminor axis: b
Foci: $F_1(-c, 0)$, $F_2(c, 0)$
Distance between the foci: $2c$
Eccentricity: e
Real numbers: A, B, C, D, E, F, t
Perimeter: L
Area: S

645. Equation of an Ellipse (Standard Form)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

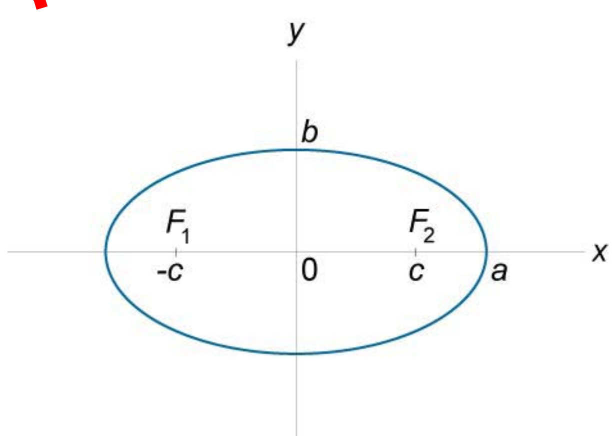
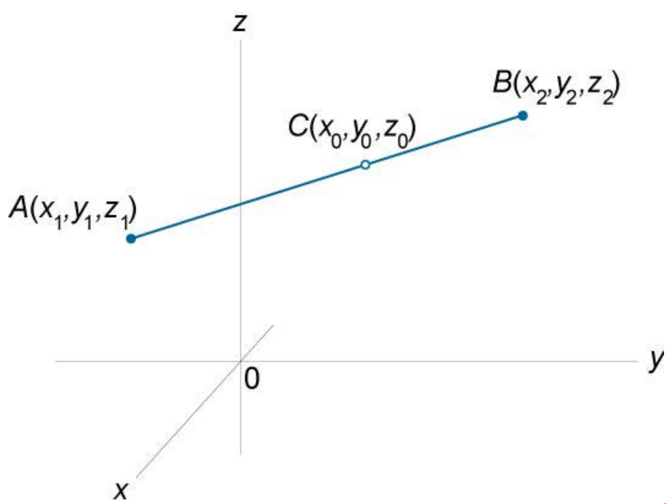
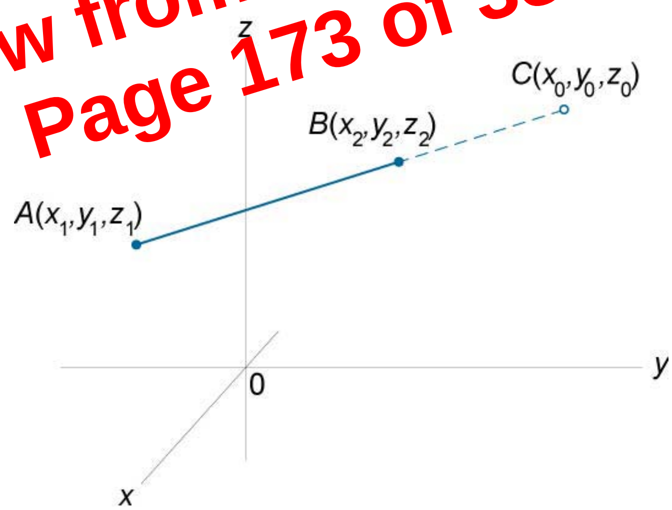


Figure 115.



$\lambda > 0$

Figure 124.



$\lambda < 0$

Figure 125.

672. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ y_0 = \frac{y_1 + y_2}{2},\ z_0 = \frac{z_1 + z_2}{2},\ \lambda = 1.$$

673. Area of a Triangle

The area of a triangle with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ is given by

$$S = \frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}.$$

674. Volume of a Tetrahedron

The volume of a tetrahedron with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, and $P_4(x_4, y_4, z_4)$ is given by

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$

or

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}.$$

Note: We choose the sign (+) or (–) so that to get a positive answer for volume.

676. Normal Vector to a Plane
The vector $\vec{n}(A, B, C)$ is normal to the plane
$Ax + By + Cz + D = 0$.

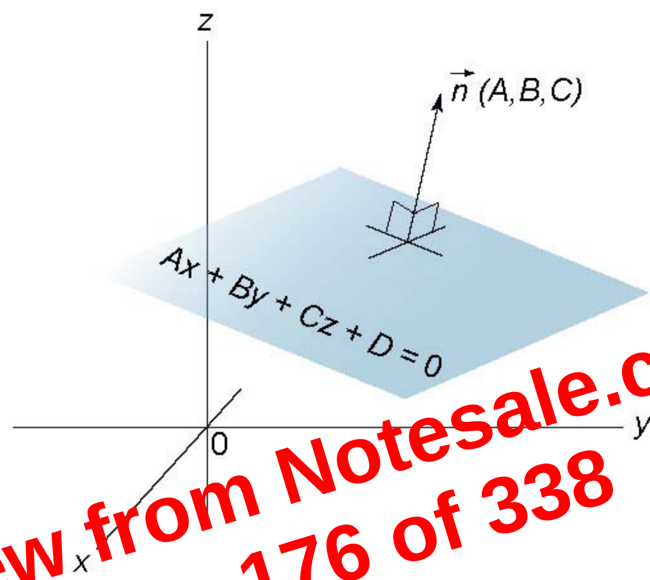


Figure 127.

677. Particular Cases of the Equation of a Plane
$Ax + By + Cz + D = 0$

If $A = 0$, the plane is parallel to the x-axis.
If $B = 0$, the plane is parallel to the y-axis.
If $C = 0$, the plane is parallel to the z-axis.
If $D = 0$, the plane lies on the origin.

If $A = B = 0$, the plane is parallel to the xy-plane.
If $B = C = 0$, the plane is parallel to the yz-plane.
If $A = C = 0$, the plane is parallel to the xz-plane.

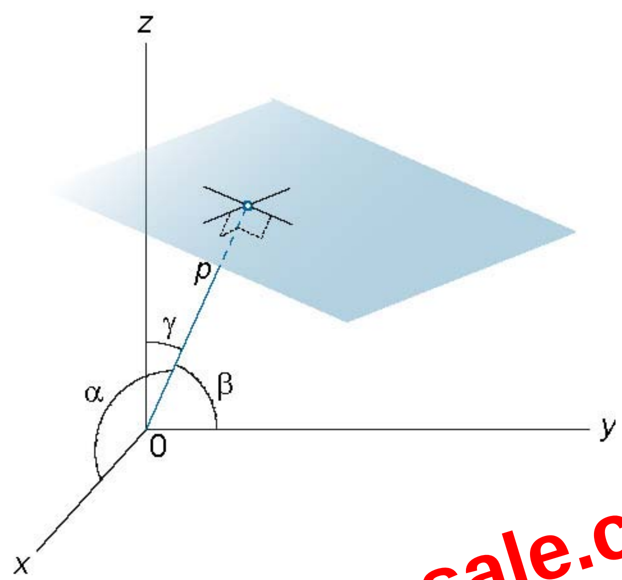


Figure 131.

682. Parametric Form

$$\begin{cases} x = x_1 + a_1 s + a_2 t, \\ y = y_1 + b_1 s + b_2 t, \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

where (x,y,z) are the coordinates of any unknown point on the line , the point $P(x_1,y_1,z_1)$ lies in the plane, the vectors (a_1,b_1,c_1) and (a_2,b_2,c_2) are parallel to the plane.

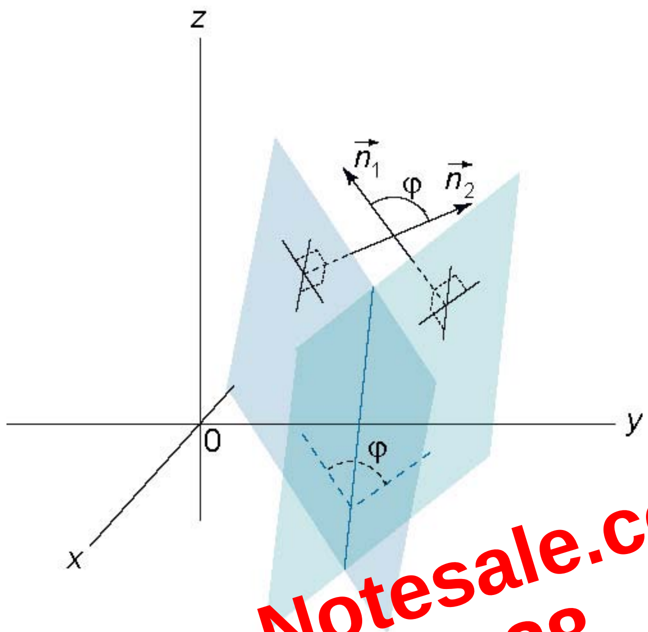


Figure 133.

684. Parallel Planes
Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel if
$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

685. Perpendicular Planes
Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if
$A_1A_2 + B_1B_2 + C_1C_2 = 0$.

686. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel To the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) (Fig.132)

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ intersect if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

697. Parallel Line and Plane

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$ are parallel if

$\vec{n}\cdot\vec{s}=0,$

or

$Aa+Bb+Cc=0.$

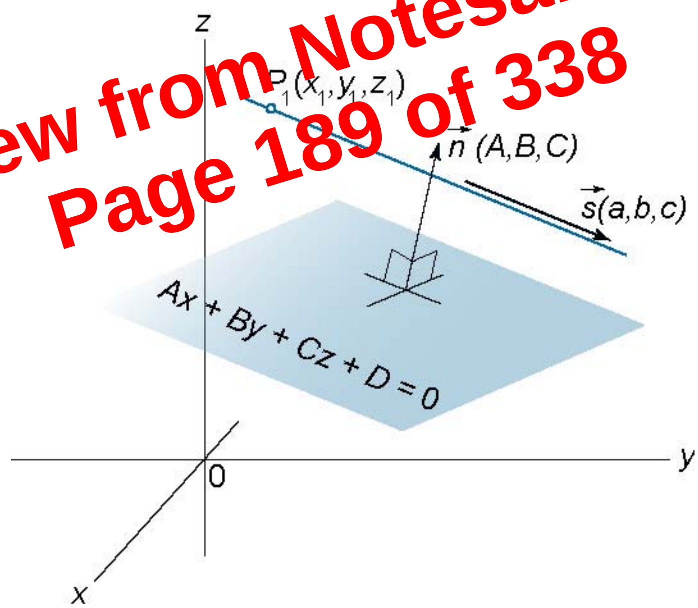


Figure 140.

699. General Quadratic Equation

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Px + 2Qy + 2Rz + D = 0$$

700. Classification of Quadric Surfaces

Case	Rank(e)	Rank(E)	Δ	k signs	Type of Surface
1	3	4	< 0	Same	Real Ellipsoid
2	3	4	> 0	Same	Imaginary Ellipsoid
3	3	4	> 0	Different	Hyperboloid of 1 Sheet
4	3	4	< 0	Different	Hyperboloid of 2 Sheets
5	3	3		Different	Real Quadric Cone
6	3	3		Same	Imaginary Quadric Cone
7	2	4	< 0	Same	Elliptic Paraboloid
8	2	4	> 0	Different	Hyperbolic Paraboloid
9	2	3		Same	Real Elliptic Cylinder
10	2	3		Same	Imaginary Elliptic Cylinder
11	2	3		Different	Hyperbolic Cylinder
12	2	2		Different	Real Intersecting Planes
13	2	2		Same	Imaginary Intersecting Planes
14	1	3			Parabolic Cylinder
15	1	2			Real Parallel Planes
16	1	2			Imaginary Parallel Planes
17	1	1			Coincident Planes

Here

$$e = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}, \; E = \begin{pmatrix} A & H & Q & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{pmatrix}, \; \Delta = \det(E),$$

k_1, k_2, k_3 are the roots of the equation,

$$\begin{vmatrix} A - x & H & G \\ H & B - x & F \\ G & F & C - x \end{vmatrix} = 0.$$

709. Real Elliptic Cylinder (Case 9)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

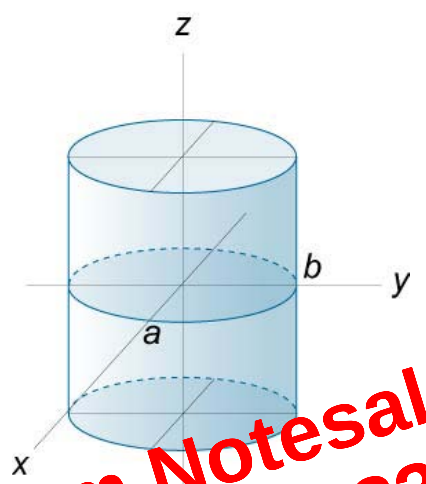


Figure 148.

710. Imaginary Elliptic Cylinder (Case 10)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

711. Hyperbolic Cylinder (Case 11)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

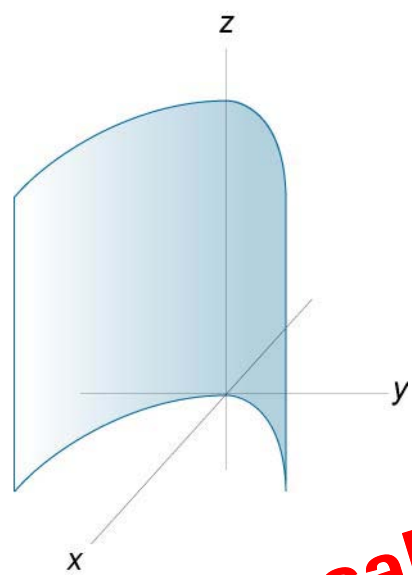


Figure 150.

715. Real Parallel Planes (Case 15)

$$\frac{x^2}{a^2} = 1$$

716. Imaginary Parallel Planes (Case 16)

$$\frac{x^2}{a^2} = -1$$

717. Coincident Planes (Case 17)

$$x^2 = 0$$

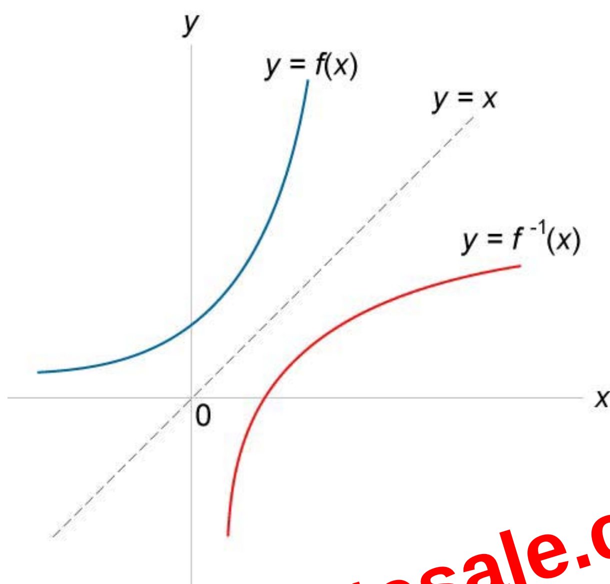


Figure 152.

727. Composite Function
$y = f(u)$, $u = g(x)$, $y = f(g(x))$ is a composite function.

728. Linear Function
$y = ax + b$, $x \in R$, $a = \tan\alpha$ is the slope of the line, b is the y-intercept.

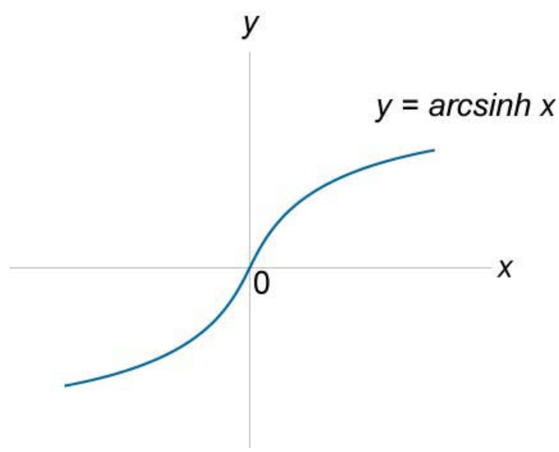


Figure 169.

744. Inverse Hyperbolic Cosine Function
$y = \operatorname{arccosh} x,\ x \in [1, \infty)$.

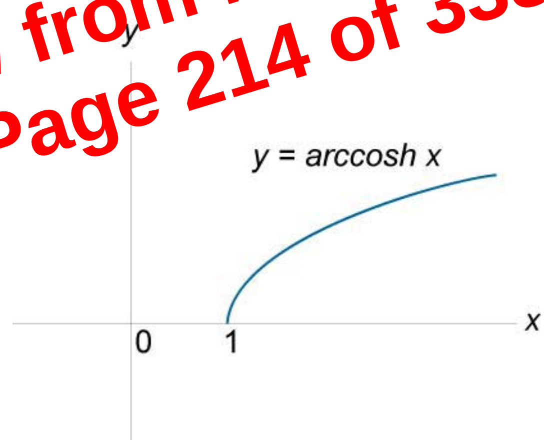


Figure 170.

745. Inverse Hyperbolic Tangent Function
$y = \operatorname{arctanh} x,\ x \in (-1, 1)$.

759. $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}=1$

760. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

761. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

762. $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}$

763. $\lim\limits_{x\to 0}a^{x}=1$

8.3 Definition and Properties of the Derivative

Functions: f, g, y, u, v
Independent variable: x
Real constant: k
Angle: α

764. $y'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$

775. $\frac{d}{dx}(C)=0$

776. $\frac{d}{dx}(x)=1$

777. $\frac{d}{dx}(ax+b)=a$

778. $\frac{d}{dx}(ax^2+bx+c)=ax+b$

779. $\frac{d}{dx}(x^n)=nx^{n-1}$

780. $\frac{d}{dx}(x^{-n})=-\frac{n}{x^{n+1}}$

781. $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

782. $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$

783. $\frac{d}{dx}(\sqrt[n]{x})=\frac{1}{n\sqrt[n]{x^{n-1}}}$

784. $\frac{d}{dx}(\ln x)=\frac{1}{x}$

785. $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$, $a>0$, $a\neq 1$.

797. $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$

798. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$

799. $\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

800. $\frac{d}{dx}(\sinh x) = \cosh x$

801. $\frac{d}{dx}(\cosh x) = \sinh x$

802. $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

803. $\frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

804. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$

805. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$

806. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$

807. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}$

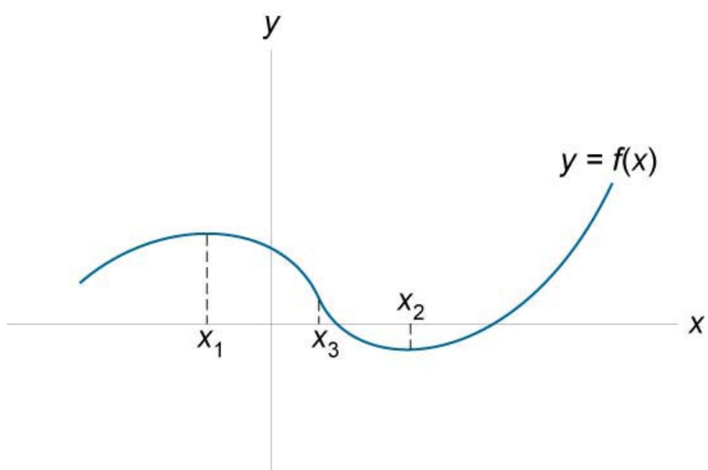


Figure 177.

829. Local extrema

A function f(x) has a local maximum at x_1 if and only if there exists some interval containing x_1 such that $f(x_1) \geq f(x)$ for all x in the interval (Fig.177).

A function f(x) has a local minimum at x_2 if and only if there exists some interval containing x_2 such that $f(x_2) \leq f(x)$ for all x in the interval (Fig.177).

830. Critical Points

A critical point on f(x) occurs at x_0 if and only if either $f'(x_0)$ is zero or the derivative doesn't exist.

831. First Derivative Test for Local Extrema.

If f(x) is increasing ($f'(x) > 0$) for all x in some interval $(a, x_1]$ and f(x) is decreasing ($f'(x) < 0$) for all x in some interval $[x_1, b)$, then f(x) has a local maximum at x_1 (Fig.177).

8.7 Differential

Functions: f, u, v
Independent variable: x
Derivative of a function: $y'(x)$, $f'(x)$
Real constant: C
Differential of function $y = f(x)$: dy
Differential of x: dx
Small change in x: Δx
Small change in y: Δy

838. $dy = y'dx$

839. $f(x + \Delta x) = f(x) + f'(x)\Delta x$

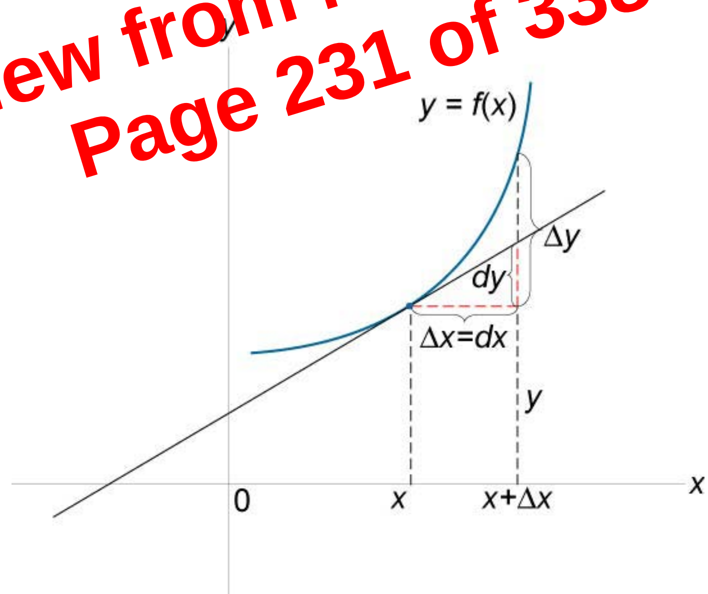


Figure 178.

880. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1.$

881. $\int \frac{dx}{x} = \ln|x| + C$

882. $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

883. $\int \frac{ax+b}{cx+d}dx = \frac{a}{c}x + \frac{bc-ad}{c^2}\ln|cx+d| + C$

884. $\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C,\ a \neq b.$

885. $\int \frac{xdx}{a+bx} = \frac{1}{b^2}\left(a+bx - a\ln|a+bx|\right) + C$

886. $\int \frac{x^2dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\right] + C$

887. $\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left|\frac{a+bx}{x}\right| + C$

888. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$

889. $\int \frac{xdx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right) + C$

945. $\int \left(x^2 - a^2\right)^{3/2} dx = -\frac{x}{8}\left(2x^2 - 5a^2\right)\sqrt{x^2 - a^2} +$

$+\frac{3a^4}{8}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$

946. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$

947. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}\left(a^2 - x^2\right)^{3/2} + C$

948. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}\left(2x^2 - a^2\right)\sqrt{a^2 - x^2} + \frac{a^4}{8}\arcsin\frac{x}{a} + C$

949. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} + a\ln\left|\frac{x}{a + \sqrt{a^2 - x^2}}\right| + C$

950. $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin\frac{x}{a} + C$

951. $\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$

952. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin\frac{x}{a} + C$

953. $\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + C$

954. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$

975. $\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + C$

976. $\int \sin x \cos^2 x \, dx = -\frac{1}{3}\cos^3 x + C$

977. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

978. $\int \tan x dx = -\ln|\cos x| + C$

979. $\int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + C = \sec x + C$

980. $\int \frac{\sin^2 x}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \sin x + C$

981. $\int \tan^2 x \, dx = \tan x - x + C$

982. $\int \cot x dx = \ln|\sin x| + C$

983. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C = -\csc x + C$

984. $\int \frac{\cos^2 x}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + \cos x + C$

985. $\int \cot^2 x \, dx = -\cot x - x + C$

986. $\int \frac{dx}{\cos x \sin x} = \ln|\tan x| + C$

1008. $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

9.6 Integrals of Exponential and Logarithmic Functions

1009. $\int e^x dx = e^x + C$

1010. $\int a^x dx = \frac{a^x}{\ln a} + C$

1011. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

1012. $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$

1013. $\int \ln x \, dx = x \ln x - x + C$

1014. $\int \frac{dx}{x \ln x} = \ln|\ln x| + C$

1015. $\int x^n \ln x \, dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right] + C$

1016. $\int e^{ax} \sin bx \, dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax} + C$

$$-\frac{n-1}{n+m}\int \sinh^{n-2} x \cosh^{m} x dx$$

1026. $\int \tanh^{n} x dx = -\frac{1}{n-1}\tanh^{n-1} x + \int \tanh^{n-2} x dx\,,\ n \neq 1\,.$

1027. $\int \coth^{n} x dx = -\frac{1}{n-1}\coth^{n-1} x + \int \coth^{n-2} x dx\,,\ n \neq 1\,.$

1028. $\int \operatorname{sech}^{n} x dx = \frac{\operatorname{sech}^{n-2} x \tanh x}{n-1} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} x dx\,,\ n \neq 1\,.$

1029. $\int \sin^{n} x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$

1030. $\int \frac{dx}{\sin^{n} x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}\,,\ n \neq 1\,.$

1031. $\int \cos^{n} x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$

1032. $\int \frac{dx}{\cos^{n} x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}\,,\ n \neq 1\,.$

1033. $\int \sin^{n} x \cos^{m} x dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m}$

$$+\frac{m-1}{n+m}\int \sin^{n} x \cos^{m-2} x dx$$

1034. $\int \sin^{n} x \cos^{m} x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m}$

1063. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \text{ if } F'(x) = f(x).$$

1064. Method of Substitution

If $x = g(t)$, then

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt,$$

where

$$c = g^{-1}(a),\ d = g^{-1}(b).$$

1065. Integration by Parts

$$\int_a^b udv = (uv)\Big|_a^b - \int_a^b vdu$$

1066. Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right]$$

- If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is also convergent,
- If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is also divergent.

1075. Absolute Convergence

If $\int_a^{\infty} |f(x)|dx$ is convergent, then the integral $\int_a^{\infty} f(x)dx$ is absolutely convergent.

1076. Discontinuous Integrand
Let $f(x)$ be a function which is continuous on the interval $[a,b)$ but is discontinuous at $x = b$. Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx$$

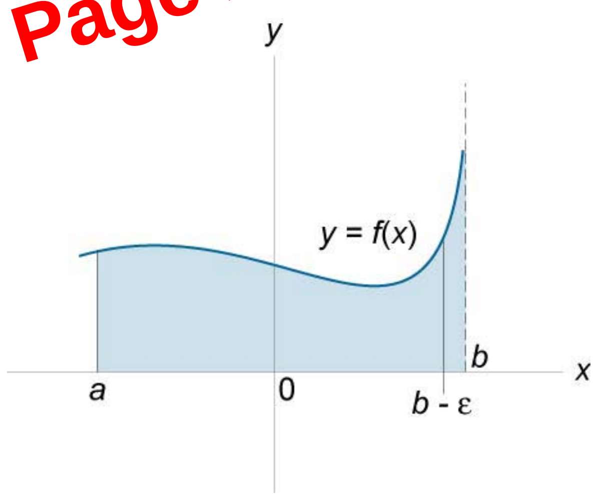


Figure 187.

$$\iint_S f(x,y)\,dA \le \iint_R f(x,y)\,dA .$$

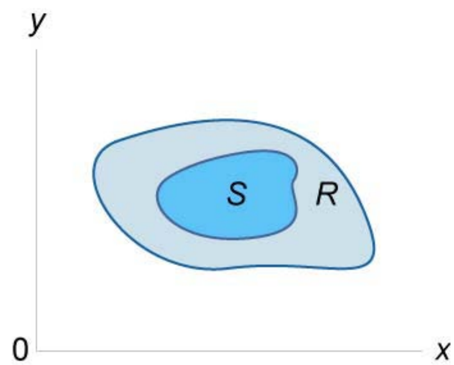


Figure 191.

1084. If $f(x,y) \ge 0$ on R and R and S are non-overlapping regions, then $\iint\limits_{R \cup S} f(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_S f(x,y)\,dA$.
Here $R \cup S$ is the union of the regions R and S.

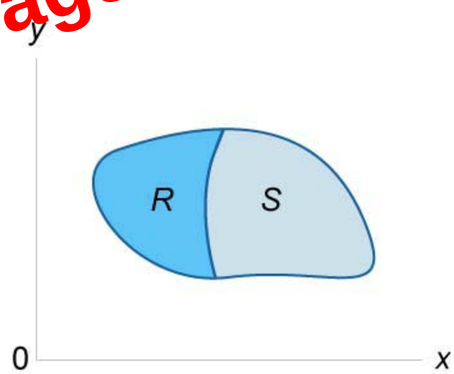


Figure 192.

1130. Length of a Curve

$$L=\int_C ds=\int_\alpha^\beta\left|\frac{d\vec{r}}{dt}(t)\right|dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt,$$

where C ia a piecewise smooth curve described by the position vector $\vec{r}(t)$, $\alpha\le t\le\beta$.

If the curve C is two-dimensional, then

$$L=\int_C ds=\int_\alpha^\beta\left|\frac{d\vec{r}}{dt}(t)\right|dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

If the curve C is the graph of a function $y=f(x)$ in the xy-plane $(a\le x\le b)$, then

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

1131. Length of a Curve in Polar Coordinates

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta,$$

where the curve C is given by the equation $r=r(\theta)$, $\alpha\le\theta\le\beta$ in polar coordinates.

1132. Mass of a Wire

$$m=\int_C\rho(x,y,z)ds,$$

where $\rho(x,y,z)$ is the mass per unit length of the wire.

If C is a curve parametrized by the vector function $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$, then the mass can be computed by the formula where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)

$$\Phi = \oiint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{F} = \rho\vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge

$$Q = \iint_S \sigma(x,y)\,dS,$$

where $\sigma(x,y)$ is the surface charge density.

1163. Gauss' Law

The electric flux through any closed surface is proportional to the charge Q enclosed by the surface.

$$\Phi = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$

where

Φ is the electric flux,

$\vec{E}$ is the magnitude of the electric field strength,

$\varepsilon_0 = 8{,}85 \times 10^{-12}\ \frac{F}{m}$ is permittivity of free space.

1172. Population Dynamics (Logistic Model)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where $P(t)$ is population at time t, k is a positive constant, M is a limiting size for the population.

The solution of the differential equation is

$P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kt}}$, where $P_0 = P(0)$ is the initial population at time $t = 0$.

10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = 0$.

The characteristic equation is

$\lambda^2 + p\lambda + q = 0$.

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, where

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\dfrac{p}{2}$, then the general solution is

$$y = (C_1 + C_2 x)e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

Chapter 11

Series

11.1 Arithmetic Series

Initial term: a_1
Nth term: a_n
Difference between successive terms: d
Number of terms in the series: n
Sum of the first n terms: S_n

1184. $a_n = a_{n-1} + d = a_{n-2} + 2d = \ldots = a_1 + (n-1)d$

1185. $a_1 + a_n = a_2 + a_{n-1} = \ldots = a_i + a_{n+1-i}$

1186. $a_i = \dfrac{a_{i-1} + a_{i+1}}{2}$

1187. $S_n = \dfrac{a_1 + a_n}{2} \cdot n = \dfrac{2a_1 + (n-1)d}{2} \cdot n$

1248. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

1249. $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

1250. $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

1260. Range of Probability Values
$0 \le P(A) \le 1$

1261. Certain Event
$P(A) = 1$

1262. Impossible Event
$P(A) = 0$

1263. Complement
$P(\overline{A}) = 1 - P(A)$

1264. Independent Events
$P(A/B) = P(A)$,
$P(B/A) = P(B)$

1265. Addition Rule for Independent Events
$P(A \cup B) = P(A) + P(B)$

1266. Multiplication Rule for Independent Events
$P(A \cap B) = P(A) \cdot P(B)$

1267. General Addition Rule
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
where
$A \cup B$ is the union of events A and B,
$A \cap B$ is the intersection of events A and B.

1268. Conditional Probability
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

1269. $P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(B/A)$