Limits & continuity in several variables: Special limits: $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$. $\sin^2\theta + \cos\theta = 1$

Showing that a limit does not exist: ① evaluate at $f(x,0)$, $f(y,0)$ any $= x$. If the limits don't equal ⇒ DNE

When solving limit problems do the following:
① Evaluate lim at pts given → if $\frac{0}{0}$ ← 0 ⇐ undefined ⇒ Squeeze thrm.
② If lim doesn't have $\frac{0}{0}$ then evaluate for $(x,0)$ & $(y,0)$ & $y=x$ or any other line specified ex: $y=2x$. If all are equal. If not ⇒ DNE

Squeeze thrm: ① Convert to polar coordinates if given x & y → $x = r\cos\theta$, $y = r\sin\theta$ ② Substitute $r\cos\theta$ for x & $r\sin\theta$ for y
③ Simplify → usually have $\cos^2\theta + \sin^2\theta = 1$ term ③ evaluate limit. For Sin & Cos bounds are always $-1 \le \cos \le 1$ or $-1 \le \sin \le 1$
④ Multiply thru by coefficient in front of Sin & Cos ⑤ evaluate bounds → $f(x)$ = bounds outer & inner. Squeeze thrm problems usually in the form $\frac{xy}{x^2+y^2}$ ← denominator

Partial derivatives: ① "freeze" one variable (keep constant) → ② take derivative of the other variables
→ Remember order doesn't matter for higher derivatives $f_{xxyy} = f_{xyxy} = f_{yyxx}$. Also, $\frac{\partial f}{\partial x} = f_x$ ← same thing
Conceptually ⇒ the partial derivatives are the slopes of the tangent lines to the vertical traces.
Can estimate partial derivatives numerically using $f_x(a,b) \approx \frac{\Delta f}{\Delta x}$ and $f_y(a,b) \approx \frac{\Delta f}{\Delta y}$

Remember - when taking derivative of fractions → its actually numerator x denominator^{-1} so the bottom part is part of equation! When you have something added together parts not involved will disappear. ex: $\frac{\partial}{\partial z}(e^{2x}+y) = xe^{2x+y}$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ Product Rule: $f(x)g(x) = f'g + fg'$ Chain Rule: $f(g(x))' = f'(g(x))\cdot g'(x)$

Power rule & chain rule: $\frac{d}{dx} g(x)^n = n(g(x))^{n-1} \cdot g'(x)$, Exponent rule: $\frac{d}{dx} e^{g(x)} = g'(x)e^{g(x)}$

General flowchart for partial derivative Problems ① Circle which variable you are taking the derivative with respect to. ② take derivative keep all other variables constant!
If two of the same variables are multiplied together make sure to take product rule!

Equation of a tangent plane & Linearization is the same thing: for Linearization: $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
Tangent plane equation is the same but says $z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
Flowchart ① evaluate original function $f(a,b)$ at pts (a,b) ② take f_x & f_y ③ evaluate f_x & f_y at pts (a,b) ④ Apply equation (plug into equation)

Linear Approximation: Two different forms: $f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k$ where $x = a+h$, $y = b+k$
another form: $\Delta f \approx f(a,b) + f(a,b)\Delta x + f_y(a,b)\Delta y$ where $\Delta f = f(x,y) - f(a,b)$ $\Delta x = x-a$, $\Delta y = y-b$

Gradient & Directional Derivative
Gradient: $\nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>$ ⇐ vector form ∇f_p = means gradient evaluated at a point p.

Chain Rule for Paths: Recall, a path is represented by: $c(t) = \langle x(t), y(t), z(t) \rangle$; $c'(t) = \langle x'(t), y'(t), z'(t) \rangle$
Flowchart for Chain rule for Paths: $\frac{d}{dt} f(c(t)) = \nabla f_{c(t)} \cdot c'(t)$ → vector form; dot product! scaler ⇒ #
① Find gradient $\nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>$ ② evaluate original functions at points given $c(t)$ ③ take derivative of $c(t)$ ⇒ $c'(t)$, ④ evaluate $c'(t)$ @ pts given ⑤ take dot product: $\nabla f(c(t)) \cdot c'(t)$ ⇐ dot product
→ for 2 variables $\frac{d}{dt} f(c(t)) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> \cdot \langle x'(t), y'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ #
③ evaluate $f_{c(t)}$

Directional derivatives: $D_u f(p) = \frac{1}{\|v\|} \nabla f_p \cdot v$ where $u = \frac{v}{\|v\|}$ (unit vector normalized)
Flowchart:
① normalize the vector $u = \frac{v}{\|v\|}$ ← unit vector; magnitude $= \sqrt{(x)^2 + (y)^2}$ Remember unit vector: $\|e v\| = \frac{1}{\|v\|} \cdot v$ → end result is a scalar
② find the gradient $\nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>$ ③ evaluate gradient at point $P(a,b)$ ⇒ ∇f_p ④ take dot product of $\nabla f \cdot u = D_u$. Also have $D_u(a,b) = \nabla f_{(a,b)} \cdot v$
*If looking for directional derivative given an angle θ ⇒ rate of change in a given direction varies with cosine (θ) between gradient and direction. From Section 12.3 angle between two non zero vectors is given by $v \cdot w = \|v\|\|w\| \cos\theta$ or $\cos\theta = \frac{v\cdot w}{\|v\|\|w\|}$

Given angle θ ⇒ $D_u f(P) = \|\nabla f(a,b)\| \cos\theta$ maximum rate of increase: $\|\nabla f_p\|$ ← magnitude $\theta = \frac{\pi}{2}$; minimum rate of decrease: $-\|\nabla f_p\|$
Flowchart ① take gradient ② evaluate at $P(a,b)$ ③ take magnitude
Conceptually: ∇f_p is normal to the level curve of f at point P
* Recall, $v \perp w$ ⇒ $v \cdot w = 0$ ← $\theta = \pi/2$ Another use of gradient: Finding normal vectors to a surface: $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$

Chain Rule (Section 14.6): To evaluate the composite... Application of chain rule... ① take gradient of ∇f ② take derivative x, y, z (given equations)
③ use chain rule to solve: $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$ ⇒ notice these are dot products. ⇒ output is scaler

Polar coordinates: use $x = r\cos\theta$, $y = r\sin\theta$; $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$ ... chain rule: $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y}$ ⇒ $\frac{\partial f}{\partial \theta} = x \cdot \frac{\partial f}{\partial y} + y\frac{\partial f}{\partial x}$

Implicit differentiation: $z = z(x,y)$ and $F(x,y,z) = 0$ Using Implicit differentiation: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ ← written in terms of x & y, where $z = z(x,y)$

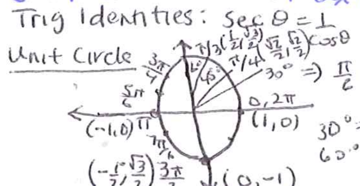
Optimization = ① Critical points ⇒ ① Take $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or DNE ② Solve for x & y
② Determine the type of critical point using the 2nd derivative test
Type of critical point depends on the sign of the discriminant: $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f^2_{xy}(a,b)$
2nd derivative test: Let $P = (a,b)$ critical point of $f(x,y)$
① if $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is local minimum ② If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is local maximum
③ if $D < 0$ then f has a Saddle Point at (a,b) ④ if $D = 0$ test is inconclusive

Saddle point when it is neither a local min nor a local max. One direction takes you uphill the other downhill
Graphically: from contour maps can determine local min
Section 14.1 Plotting 2-D Graphs & contour maps & Level curves
Contour maps - level curves close together ⇒ graph is steep. Small change from one level to another. ⇒ large change in height. Flatter part of curve → curves are further apart.
← Local min increasing in all directions

Saddle pt - increasing in some directions while decreasing in others.

When reading a map → ① Altitude doesn't change → same along the same level curve. ② Altitude increases or decreases by spacing level m (contour level) going from one level to the next. ③ Spacing of level → steepness closer together where graph is steeper.
Average rate of change: $= \frac{\Delta \text{altitude}}{\Delta \text{horizontal}}$ $\frac{\Delta\text{altitude } (y \text{ direction})}{\Delta\text{horizontal } (x \text{ direction})}$
A path of steepest ascent = everywhere along the way points in the steepest direction & moves to next level along the shortest possible segment. altitude increases most rapidly
Vertical trace - intersects graph with vertical plane $x = a$ or $y = b$
↳ freeze one variable by setting it = 0 constant & plotting the other
horizontal trace: ① Set $f(x,y) = c$, ② Solve for y in terms of c (intersection of $f(x,y)$ at $z = c$)
Partial derivatives are the slopes of the tangent lines to the vertical traces
Partial derivatives: know: $\frac{\partial^2}{\partial x^2} f = f_{xx}$ $\frac{\partial^2}{\partial y \partial x} f = f_{xy}$ $\frac{\partial^2}{\partial x \partial y} = f_{yx}$ ↑ 1st derivative ↑ 1st
Tangent plane → plane containing tangent lines to two vertical trace curves through P.
Directional derivative → slope of tangent line to the curve when graph intersects the plane
↳ At saddle pt, a graph can take a variety of different shapes. ∇f_p is normal to level surface

Important things to know
① When you have two pts given $P(a,b)$ $Q = (c,d)$ ⇒ take $\overrightarrow{PQ} = (c-a, d-b)$ to get vector
② dot product: $v \cdot w = a_1a_2 + b_1b_2$ ⇒ scaler (end result #)
③ Equation of a Plane: $ax + by + cz = d$ given $P = (x_0, y_0, z_0)$ $n = (a,b,c)$ $d = n(x_0y_0z_0) = ax_0 + by_0 + cz_0$
Ex: $(\cos 4\theta)' = 4\cos\theta\sin\theta$ raised to the 4th power; tangent line → ∇f_p; level curve
Trig identities: $\sec\theta = \frac{1}{\cos\theta}$
Unit Circle



$30° \Rightarrow \frac{\pi}{6}\left(\frac{\sqrt3}{2}, \frac{1}{2}\right)$; $30° \Rightarrow \frac{\pi}{6} \Rightarrow \frac{\sqrt3}{2}, \frac{1}{2}$; $60° = \frac{\pi}{3} \Rightarrow \frac{1}{2}, \frac{\sqrt3}{2}$

double angle formula: $\sin 2x = 2\sin x\cos x$, $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$
Log rule: $\log x^r = r\log x$ ⇐ Same for Ln; ex. $\ln^2 \Rightarrow 2\ln x$
$\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\frac{d}{dx}(a^x) = \ln a(a^x)$, $\frac{d}{dx}\ln x = \frac{1}{x}$ Examples $\frac{d}{dx}(\cos 4\theta) = -\sin(4\theta)\cdot 4$; $e^{\cos x} = e^{\cos t}\cdot \sin x$