and $$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[x^{y-1}\cdot(1+y\log x)\right] \qquad \ldots(1)$$

Again $$\frac{\partial u}{\partial x} = x^{y-1} \text{ and } \frac{\partial^2 u}{\partial y \partial x} = 1\cdot x^{y-1} + y(x^{y-1}\log x) = x^{y-1}(1+y\log x) \qquad \ldots(2)$$

Using (2),

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left[x^{y-1}(1+y\log x)\right] \qquad \ldots(3)$$

From (1) and (3) follows the result.

Example 9: If $v = \log(x^2+y^2+z^2)$, prove that $(x^2+y^2+z^2)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = 2$

Solution: Given $v = \log(x^2+y^2+z^2)$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial x}\right)_{y,z} = \frac{1}{(x^2+y^2+z^2)}2x \qquad \ldots(1)$$

And $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = 2\left[\frac{(x^2+y^2+z^2)\cdot 1 - x\cdot 2x}{(x^2+y^2+z^2)^2}\right] = 2\left[\frac{(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}\right] \qquad \ldots(2)$$

Similarly $$\frac{\partial^2 v}{\partial y^2} = 2\left[\frac{(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}\right] \qquad \ldots(3)$$

and $$\frac{\partial^2 v}{\partial z^2} = 2\left[\frac{(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}\right] \qquad \ldots(4)$$

Adding (2), (3) and (4), we get

$$\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = 2\left[\frac{(x^2+y^2-x^2)+(x^2+z^2-y^2)+(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}\right]$$

$$\Rightarrow \quad \left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = \frac{2}{(x^2+y^2+z^2)}$$

or $$(x^2+y^2+z^2)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = 2$$

Hence the result.

Example 10: If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$ **[UP Tech, 2003]**

On using results given by (7), (8), (9) into equation (6), we get

$$\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=f''(r)\left(\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2+\left(\frac{z}{r}\right)^2\right)+f'(r)\left(\frac{r^2-x^2+r^2-y^2+r^2-z^2}{r^3}\right)$$

$$=f''(r)\left(\frac{x^2+y^2+z^2}{r^2}\right)+f'(r)\left(\frac{3r^2-r^2}{r^3}\right)=f''(r)+\frac{2}{r}f'(r)$$

ASSIGNMENT 2

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that (*i*) $\dfrac{\partial r}{\partial x}=\dfrac{\partial x}{\partial r}$ (*ii*) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta}=r\dfrac{\partial \theta}{\partial x}$, (*iii*) $\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2=1$.

2. If $u = f(r)$ and $\left.\begin{matrix} x=r\cos\theta \\ y=r\sin\theta\end{matrix}\right\}$, prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f''(r)+\dfrac{1}{r}f'(r)$. **[Burdwan, 2003]**

3. If $x=\dfrac{\cos\theta}{u}, y=\dfrac{\sin\theta}{u}$, show that $\left(\dfrac{\partial x}{\partial u}\right)_\theta\left(\dfrac{\partial u}{\partial x}\right)_y+\left(\dfrac{\partial y}{\partial u}\right)_\theta\left(\dfrac{\partial u}{\partial y}\right)_x=1$.

4.4 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

Homogeneous Function

An expression of the form $(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n)$ in which all the terms are of degree n, is called a homogeneous function of degree n.

The above expression mathematically may be rewritten as

$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \dots + a_n(y/x)^n]$ or more precisely $x^n\phi(y/x)$,

where $\phi(y/x)$ is a polynomial of degree n in (y/x).

Thus, any function $f(x, y)$ which is expressible as either $x^n\phi(y/x)$ or $y^n\psi(x/y)$ is called a homogeneous function of degree n in x and y.

E.g. (*i*) $x^3\tan(y/x)$ is a homogeneous function of degree 3 in x and y.

(*ii*) $\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}=\dfrac{x^{1/3}\left\{1+\left(\dfrac{y}{x}\right)^{1/3}\right\}}{x^{1/2}\left\{1+\left(\dfrac{y}{x}\right)^{1/2}\right\}}=x^{1/3-1/2}\phi(y/x)=x^{-1/6}\phi(y/x)$ is a homogeneous function of degree $-1/6$ in x and y.

In general, a function $f(x_1, x_2, x_3, \dots)$ is said to be a homogeneous function of degree n in $(x_1, x_2, \dots, x_n)$ if it is expressible in the form as $x_1^n\,\phi\left(\dfrac{x_2}{x_1},\dfrac{x_3}{x_1},\dots,\dfrac{x_n}{x_1}\right)$.

Euler's Theorem

If u be homogeneous function of degree n in x, y and has continuous first derivatives then

$x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu.$ **[KUK, 2004]**

Now multiply (3) by x and (4) by y and, then add the two, we get

$$x\frac{\partial^2 Z_1}{\partial x^2}+\left(x\frac{\partial Z_1}{\partial x}+y\frac{\partial Z_1}{\partial y}\right)+2xy\frac{\partial^2 Z_1}{\partial y\partial x}+y^2\frac{\partial Z_1}{\partial y^2}=n\left[x\frac{\partial Z_1}{\partial x}+y\frac{\partial Z_1}{\partial y}\right]$$

$$\Rightarrow \quad x\frac{\partial^2 Z_1}{\partial x^2}+\left(x\frac{\partial Z_1}{\partial x}+y\frac{\partial Z_1}{\partial y}\right)+2xy\frac{\partial^2 Z}{\partial y\partial x}+y^2\frac{\partial^2 Z_1}{\partial y^2}=n\,nZ_1=n^2Z_1 \qquad \ldots(5)$$

Similarly on differentiating (2) partially with respect to x (when y constant) and with respect to y (when x constant),

$$x\frac{\partial^2 Z_2}{\partial x^2}+\frac{\partial Z_2}{\partial x}+y\frac{\partial^2 Z_2}{\partial y\partial x}=-n\frac{\partial Z_2}{\partial x} \qquad \ldots(6)$$

and

$$x\frac{\partial^2 Z_2}{\partial x\partial y}+\frac{\partial Z_2}{\partial y}+y\frac{\partial^2 Z_2}{\partial y^2}=-n\frac{\partial Z_2}{\partial y} \qquad \ldots(7)$$

On multiplying (6) by x and (7) by y and, then adding the two

$$x^2\frac{\partial^2 Z_2}{\partial x^2}+\left(x\frac{\partial Z_2}{\partial x}+y\frac{\partial Z_2}{\partial y}\right)+2xy\frac{\partial^2 Z_2}{\partial x\partial y}+y^2\frac{\partial^2 Z_2}{\partial y^2}=-n\left[x\frac{\partial Z_2}{\partial x}+y\frac{\partial Z_2}{\partial y}\right]=-n(-n)Z_2=n^2Z_2 \qquad \ldots(8)$$

On adding (5) and (8), we have

$$x^2\frac{\partial^2 Z}{\partial x^2}+2xy\frac{\partial^2 Z}{\partial x\partial y}+y^2\frac{\partial^2 Z}{\partial y^2}+\left(x\frac{\partial Z}{\partial x}+y\frac{\partial Z}{\partial y}\right)=n^2Z$$

Example 25: If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=2u$

[GJU, 2005]

Solution: $u = u_1 + u_2$, where $u_1 = x^2\tan^{-1}(y/x)$, $u_2 = -y^2\tan^{-1}(x/y)$ are both homogeneous function of degree 2 in x and y.

$\therefore$ By Euler's Theorem, $x\dfrac{\partial u_1}{\partial x}+y\dfrac{\partial u_2}{\partial y}=2\cdot u_1 \qquad \ldots(1)$

and

$$x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=2\cdot u_2 \qquad \ldots(2)$$

Differentiating (1) partially with respect to x and y

$$x\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial u_1}{\partial x}+y\frac{\partial^2 u_1}{\partial x\partial y}=2\frac{\partial u_1}{\partial x} \qquad \ldots(3)$$

and

$$x\frac{\partial^2 u_1}{\partial y\partial x}+\frac{\partial u_1}{\partial y}+y\frac{\partial^2 u_1}{\partial y^2}=2\frac{\partial u_1}{\partial y} \qquad \ldots(4)$$

Now multiply (3) by x, (4) by y and, add the two

$$\left(x^2\frac{\partial^2 u_1}{\partial x^2}+x\frac{\partial u_1}{\partial x}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y\frac{\partial u_1}{\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2}\right)=2\left(x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}\right)$$

$$\Rightarrow \qquad \left(x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2}\right)=\left(x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}\right)=2u_1, \quad \text{(using (1))}$$

Similarly $\left(x\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2}\right)=2u_2,$ (using (2))

On adding the two, we get the desired result.

Example 26: If u is a homogeneous function of nth degree in x, y, z, prove that

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$$ **[NIT Kurukshetra, 2006]**

Solution: u is a homogeneous function of degree n in x, y and z

Let $u=x^n f\left(\frac{y}{x},\frac{z}{x}\right)$ or $u = x^n f(s, t)$ with $\frac{y}{x}=s$ and $\frac{z}{x}=t$ ...(1)

Differentiating (1) partially with respect to x

$$\frac{\partial u}{\partial x}=n\,x^{n-1}f(s,t)+x^n\left[\frac{\partial f}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial x}\right]$$

$$\frac{\partial u}{\partial x}=n\,x^{n-1}f(s,t)+x^n\left[\frac{\partial f}{\partial s}\left(-\frac{y}{x^2}\right)+\frac{\partial f}{\partial t}\left(-\frac{z}{x^2}\right)\right] \quad \text{using (1)}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x}=n\cdot x^{n-1}f(s,t)-x^{n-2}\left[y\frac{\partial f}{\partial s}+z\frac{\partial f}{\partial t}\right] \qquad ...(2)$$

On multiplying (2) by x throughout, we get

$$x\frac{\partial u}{\partial x}=n\,x^n f(s,t)-x^{n-1}\left[y\frac{\partial f}{\partial s}+z\frac{\partial f}{\partial t}\right] \qquad ...(3)$$

Now on differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y}=x^n\left[\frac{\partial f}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial y}\right]$$

$$\Rightarrow \qquad \frac{\partial u}{\partial y}=x^n\left[\frac{\partial f}{\partial s}\left(\frac{1}{x}\right)+\frac{\partial f}{\partial t}(0)\right]=x^{n-1}\frac{\partial f}{\partial s} \qquad ...(4)$$

On multiplying (4) by y throughout, we get

$$y\frac{\partial u}{\partial y}=x^{n-1}\cdot y\frac{\partial f}{\partial s} \qquad ...(5)$$

Similarly, on differentiating equation (1) with respect to z, we get

$$\frac{\partial u}{\partial z}=x^n\left[\frac{\partial f}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial z}\right]$$

(i) If we take y = constant implying $dy = 0$, (5) gives

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial z}dz=0 \quad \text{or} \quad \left(\frac{dz}{dx}\right)_y=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial z}\right)} \qquad \ldots(5a)$$

(ii) If we take x = constant implying $dx = 0$, (5) gives

$$\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \quad \text{or} \quad \left(\frac{dz}{dy}\right)_x=-\frac{\left(\frac{\partial f}{\partial y}\right)}{\left(\frac{\partial f}{\partial z}\right)} \qquad \ldots(5b)$$

(iii) If we take z = constant implying $dz = 0$, (5) gives

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \quad \text{or} \quad \left(\frac{dx}{dy}\right)_z=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} \qquad \ldots(5c)$$

Multiplying (5a), (5b), (5c), we get

$$\left(\frac{dx}{dy}\right)_z\left(\frac{dy}{dz}\right)_x\left(\frac{dz}{dx}\right)_y=-1 \quad \text{or} \quad \left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1 \qquad \ldots(5d)$$

Example 27: If $u = \sin^{-1}(x - y)$, $x = 3t$ and $y = 4t^3$, show that $\frac{du}{dt} = 3(1 - t^2)^{\frac{-1}{2}}$ [PTU, 2005]

Solution: Given, $u = \sin^{-1}(x - y)$, so that $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$ $\qquad \ldots(1)$

Now, $$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-x^2-y^2+2xy}}=\frac{1}{\sqrt{1-(3t)^2-(4t^3)^2+2(3t)(4t^3)}}$$

$$=\frac{1}{\sqrt{(1-4t^2)^2(1-t^2)}}=\frac{1}{(1-4t^2)\sqrt{(1-t^2)}}, \qquad \ldots(2)$$

$$\frac{\partial y}{\partial y}=\frac{1}{\sqrt{1-x^2-y^2+2xy}}=\frac{1}{(1-4t^2)\sqrt{1-t^2}}, \qquad \ldots(3)$$

$$\frac{dx}{dt}=3,\ \frac{dy}{dt}=12t^2. \ \text{(from given)} \qquad \ldots(4)$$

On substituting values of u_x, u_y, $\frac{dx}{dt}$, $\frac{dy}{dt}$ in (1), we get

$$\frac{du}{dt}=\frac{1}{(1-4t^2)\sqrt{1-t^2}}[3-12t^2]\equiv\frac{3(1-4t^2)}{(1-4t^2)(1-t^2)^{\frac{1}{2}}}=3(1-t^2)^{-\frac{1}{2}}.$$

Example 28: Find the total differential coefficient of x^2y with respect to x when x and y are connected by the relation $x^2 + xy + y^2 = 1$. [KUK, 2008; NIT Kurukshetra, 2008]

Solution: Let $z = x^2y$, then in this problem we need to find $\frac{dz}{dx}$.

Example 42: If $u = f(x, y)$ and $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\},$ prove that

(*i*) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2,$$

(*ii*) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$ **[Pb. Univ, 2002]**

Solution: This example is an another way of understanding the previous example. In the previous example we have just proved that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms to $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

Means the expression, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ which is in cartisian coordinate system has value $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial \theta}$ in (r, θ) system.

In the previous example, under equation (3), we had

$$\frac{\partial u}{\partial x} = \left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \text{ and } \frac{\partial u}{\partial y} = \left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

On squaring and adding these two results, we get

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)^2 + \left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left(\cos^2\theta\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\sin^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}\right)$$

$$+\left(\sin^2\theta\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}\right)$$

$$= (\cos^2\theta + \sin^2\theta)\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}(\cos^2\theta + \sin^2\theta)\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

Example 43: Given that $u(x, y, z)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\frac{\partial u}{\partial \theta}$ and $\frac{\partial u}{\partial \phi}$. **[NIT Kurukshetra, 2008]**

Solution: Here by change of variable, we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} \quad \ldots(1)$$

Now $$\left.\begin{matrix} x = r\cos\theta\cos\phi, \\ y = r\cos\theta\sin\phi, \\ z = r\sin\theta \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \frac{\partial x}{\partial \theta} = -r\sin\theta\cos\phi, \\ \frac{\partial y}{\partial \theta} = -r\sin\theta\sin\phi, \\ \frac{\partial z}{\partial \theta} = r\cos\theta \end{matrix}\right\} \quad \ldots(2)$$

Similarly $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v}\right)$$

or $$\frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2 \theta}{\partial u^2} - 4yx\frac{\partial^2 \theta}{\partial v \partial u} - 4xy\frac{\partial^2 \theta}{\partial u \partial v} + 4x^2\frac{\partial^2 \theta}{\partial v^2} \qquad \ldots(7)$$

Add (6) and (7), we get the desired result

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

ASSIGNMENT 5

1. If $u = f(r, s)$, $r = x + at$, $s = y + bt$, and x, y, t are independent variables, show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$.

2. If $u = F(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

3. If $u = f(r, s, t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ **[Karnataka, 2006]**

4. If $x = u + v + w$, $y = uw + vw + uv$ and F is a function of x, y, z, show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

5. If $x = r\cos\theta$, $y = r\sin\theta$, $z = f(x, y)$, prove that $\frac{\partial^2}{\partial x \partial y}(r^n \cos n\theta) = -n(n-1)r^{n-2}\sin(n-2)\theta$

$$\left[\text{Hint: Follow Ex. } \frac{\partial}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right), \ \frac{\partial}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\right]$$

6. If $x = e^r\cos\theta$, $y = e^r\sin\theta$; prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right)$

$$\left[\text{Hint: Use } \frac{\partial}{\partial r} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \ \frac{\partial}{\partial \theta} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\right]$$

7. If z is a function of x and y and $u = lx + my$ and $v = ly - mx$, then prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

8. If $w = u(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$ and $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$, show that

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right].$$

9. If $u = f(x, y, z)$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2$$

[**Hint:** Follow Example 42 (*ii*), Section 4.6]

Further the expression (1) is valid for u being function of any number of variables say x, y, z, t, … then,

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z + \ldots \text{ approximately} \qquad \ldots(2)$$

Note: For percentage error, it is always advisable to take log on both sides of the governing equation first and then proceed as usually.

Example 52: How sensitive is the volume $V = \pi r^2 h$ of a right circular cylinder to small changes in its radius and height near the point $(r_0, h_0) = (1, 3)$?

Solution: By increment method, we get

$$\Delta V \approx \left(\frac{\partial V}{\partial r}\right)_{(r_0, h_0)} \Delta r + \left(\frac{\partial V}{\partial h}\right)_{(r_0, h_0)} \Delta h$$

$$= V_r(1, 3)\Delta r + V_h(1, 3)\Delta h$$

$$= (2\pi rh)_{(1,3)}\, \Delta r + (\pi r^2)_{(1,3)} \cdot \Delta h$$

$$= 6\pi \cdot \Delta r + \pi \cdot \Delta h$$

The above result shows that a one-unit change in r will change V nearly by 6π units and a one-unit change in h will change V nearly by 6π units. Therefore, the volume of a cylinder with radius $r = 1$ and height $h = 3$ is nearly 6 times as sensitive to small change in r as it is to small change in h Fig. 4.3 (i).

In contrast, if value of r and h are reversed, so that $r = 3$ and $h = 1$, then $\Delta V \approx 6\pi \cdot \Delta r + 9\pi \cdot \Delta h$. The volume is now more sensitive to small change in h to that it is to change in r. Thus the sensitivity to change depends not only on the increment but also on the relative size of r and h (See Fig. 4.3 (ii)).

4.3. (i) 4.3. (ii)

Fig. 4.3

Example 53: How sensitive is the change in $V = \pi r^2 h$ related to the relative change in r and h? How are the percentage changes related?

Solution: By error approximation, $V = \pi r^2 h$ gives

$$\Delta V \approx V_r \cdot \Delta r + V_h \cdot \Delta h = 2\pi rh \cdot \Delta r + \pi r^2 \Delta h \qquad \ldots(1)$$

or

$$\frac{\Delta V}{V} \approx \frac{2\pi rh}{\pi r^2 h}\Delta r + \frac{\pi r^2}{\pi r^2 h}\Delta h = 2\frac{\Delta r}{r} + \frac{\Delta h}{h} \qquad \ldots(2)$$

Clearly the relative change in V is the relative change in h plus two times the relative change in r.

Further,

$$\left(\frac{\Delta V}{V} \times 100\right) \approx 2\left(\frac{\Delta r}{r} \times 100\right) + \left(\frac{\Delta h}{h} \times 100\right) \qquad \ldots(3)$$

indicates that the percentage change in V is the percentage change in h plus two times the percentage change in r.

Solution: Given,

$$l \text{ is decreased by } 2\% \quad \text{i.e.} \quad \frac{\delta l}{l}\times 100 = -2$$

$$r \text{ is increased by } 2\% \quad \text{i.e.} \quad \frac{\delta r}{r}\times 100 = 2$$

$$t \text{ is increased by } 1.5\% \quad \text{i.e.} \quad \frac{\delta t}{t}\times 100 = 1.5 = \frac{3}{2}$$

Torsional rigidity, $N(l,t,r) = \dfrac{8\pi I\, l}{t^2 r^4}$

implies $\quad \log N = \log(8\pi I) + \log l - 2\log t - 4\log r \qquad \ldots(1)$

Taking differentials on both sides,

$$\frac{\delta N}{N} = \frac{\delta l}{l} - 2\frac{\delta t}{t} - 4\frac{\delta r}{r}$$

or
$$\left(\frac{\delta N}{N}\times 100\right) = \left(\frac{\delta l}{l}\times 100\right) - 2\left(\frac{\delta t}{t}\times 100\right) - 4\left(\frac{\delta r}{r}\times 100\right)$$

$$= -2 - 2\times\frac{3}{2} - 4\times 2 = -13$$

Hence N diminishes by 13% if with the above given percentage changes in l, r, t.

Example 62: At a distance of 50 metres from the foot of the tower the elevation of its top is 30°. If the possible errors in measuring the distance and elevation are 2 cm and 0.05 degrees, find the approximate error in calculating the height. [NITK, 2002; UPTech, 2004]

Solution: The given problem with h as height of the top (point A) from the bottom (point B) and α the elevation of the top with the ground is $\alpha°$ as explained in Fig. 4.5. From the figure, it is apparant that height h is a function of the elevation, α and the distance of point of elevation from bottom, x, i.e.

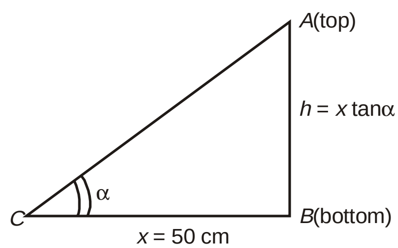

Fig. 4.5

$$h(x, \alpha) = x\tan\alpha \qquad \ldots(1)$$

so that $\quad \delta h = h_x \delta x + h_\alpha \delta\alpha = \tan\alpha\,\delta x + x\sec^2\alpha\,\delta\alpha \qquad \ldots(2)$

For given $x = 50$, $\delta x = 2 \text{ cm} = \dfrac{2}{100}$ mt; $\alpha = 30°$, $\delta\alpha = 0.05° = \dfrac{5}{100}\dfrac{\pi}{180}$ radians,

$$\delta h = (\tan 30°)\cdot\frac{2}{100} + 50\cdot\sec^2 30°\cdot\frac{5}{100}\frac{\pi}{180} = \frac{1}{\sqrt{3}}\frac{2}{100} + 50\left(\frac{2}{\sqrt{3}}\right)^2\frac{5}{100}\cdot\frac{\pi}{180}$$

$$= 0.0116 + 0.0582 = 0.0698 \simeq 0.07 \text{ mts.} = 7 \text{ cms.}$$

Example 63: If the sides and angles of a triangle ABC vary in such a way that its circum-radius remains constant. Prove that $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0$

$$R_n = \frac{1}{\lfloor\underline{n+1}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k),\ 0<\theta<1 \qquad \ldots(2)$$

If this remainder $R_n \to 0$ as $h \to \infty$, then the Taylor's Theorem becomes Taylor's Infinite series as:

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{\lfloor\underline{2}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{2} f(a,b) + \ldots\ldots\infty \qquad \ldots(3)$$

Proof: Let $x = a + ht$, $y = b + kt$, where t is the parameter which takes the values in the interval [0, 1]. Define a function

$$F(t) = f(x, y) = f(a + ht, b + kt) \qquad \ldots(4)$$

Then by chain rule on (1),

$$\frac{d}{dt}F(t) = F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)$$

Precisely,

$$\left.\begin{aligned} F'(t) &= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right) f,\\ F''(t) &= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{2} f,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ F^{n+1}(t) &= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f \end{aligned}\right\} \qquad \ldots(5)$$

On using Taylor's Theorem for functions of one variable, when $t = 1$, $a = 0$, we obtain

$$F(1) = F(0) + F'(0) + \ldots + \frac{1}{\lfloor\underline{n}}F^{n}(0) + \frac{1}{\lfloor\underline{n+1}}F^{n+1}(0) \qquad \ldots(6)$$

where $\quad F(1) = f(a+h, b+k)$

$$\left.\begin{aligned} F(0) &= f(a, b)\\ F'(0) &= f\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right) f(a, b)\\ F''(0) &= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{2} f(a, b)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ F^{n+1}(0) &= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k),\ 0<\theta<1 \end{aligned}\right\} \qquad \ldots(7)$$

With above values of $F(1)$, $F(0)$, $F'(0)$, …, $F^{(n+1)}(0)$, in (6), we get the Taylor's Theorem for functions of two variables given by equations (1) and (2) in the statement.

$$\therefore \qquad f(x,y) = \log\left[(1.03)^{\frac{1}{3}} + (0.98)^{\frac{1}{4}} - 1\right] \simeq f(1,1) + \left[0.03\, f_x(1,1) + (-0.02)\, f_y(1,1)\right] + \ldots$$

$$= 0 + \left[0.03\left(\frac{1}{3}\right) - 0.02\left(\frac{1}{4}\right)\right] = 0.005 \text{ approx.}$$

ASSIGNMENT 8

1. Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's Theorem.

2. Expand $f(x, y) = \sin xy$ in powers of $(x - 1)$ and $\left(y - \frac{\pi}{2}\right)$ upto the second degree term.

3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.

4. Expand $\sin x \sin y$ in powers of x and y as far as terms of third degree.

5. Expand $e^x \cdot \log(1 + y)$ in powers of x and y upto terms of third degree. **[PTU, 2009]**

4.10 MAXIMA-MINIMA OF TWO FUNCTIONS

Definition: A function $f(x, y)$ of two variables is said to be **maximum** at (a, b) if $\boldsymbol{f(a + h, b + k) - f(a, b) < 0}$ for sufficiently small positive or negative values of h and k, and **minimum** if $\boldsymbol{f(a + h, b + k) - f(a, b) > 0}$.

In other words, if $f(a + h, b + k) - f(a, b) = \Delta$ is negative for small values of h, k, then $f(a, b)$ is a maximum and if $f(a + h, b + k) - f(a, b) = \Delta$ is positive for small values of h, k, then $f(a, b)$ is minimum.

The points at which maximum or minimum values occur are also known as points of extrema of critical points and the maximum and minimum values taken together are **extreme values** of the function.

Observations:

1. A function $f(x, y)$ may also attain its extreme values on the boundary.
2. The maxima-minima so defined are local relative maxima or local relative minima. Thus, a maximum value may not be the greatest and minimum may not be the least of all the values of the function in any finite region.
3. The greatest and smallest values attained by a function over the entire region including the boundary are called **global (absolute) maximum** and **global (absolute) minimum** values of the function.
   E.g. for a function $z = f(x, y)$ say representing a dom, maximum value of z occurs at the top from where surface descends in all directiosns. If $z = f(x, y)$ represents the equation of a bowl, minimum is attained at the bottom from where surface ascends in all directions. Otherwise, a maximum or minimum value may form a ridge such that the surface ascends or descends in all directions.
   Besides, there are points on the surfaces, from where surface rises for displacement in certain directions and fall for displacement in the other directions, called **saddle points**.

Necessary Conditions for Having Extremum: $\boldsymbol{f_x(a, b) = 0 = f_y(a, b)}$.

By Taylor's Theorem,

$$f(a + h, b + k) = f(a, b) + \left(h\, f_x(a, b) + k\, f_y(a, b)\right) + \frac{1}{\underline{|2}}\left(h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)\right) + \ldots \qquad \ldots(1)$$

4.11 LAGRANGE'S METHOD OF UNDETERMINED MULIPLIERS: CONSTRAINED MAXIMA-MINIMA

In many engineering and science problem it is desired to find extrema of a function of several variables that are not all independent but are connected to one another by certain conditions. In such cases, generally the conventional method becomes either very complex or impracticable, then we employ an alternate method which is very easy in its approach, called Lagrange's method of undetermined multipliers. Another name of it viz. constrained maxima-minima is very obvious as here the variables involved in the function of which extreme values are to be obtained are linked to each other by certain relations which are to be taken care in the process of finding extreme values.

Illustration: Say, if we want to find the Maximum and Minimum values of

$$u = f(x, y, z) \qquad \ldots(1)$$

Consider function of three variables x, y, z which are connected by an implicit relation,

$$\phi(x, y, z) = 0 \qquad \ldots(2)$$

For function u to possess stationary values, it is necessary that

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0 \qquad \ldots(3)$$

$$\therefore \qquad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du(x, y, z) = 0 \qquad \ldots(4)$$

Also from (2),

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0 \qquad \ldots(5)$$

We see that (3) + λ(4) results in

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0 \qquad \ldots(6)$$

but this will hold true only if

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} &= 0, \quad (i) \\ \frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} &= 0, \quad (ii) \\ \frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} &= 0, \quad (iii) \end{aligned}\right\} \qquad \ldots(7)$$

whence these three equation of (7) taken together with (2) will determine those x, y, z and λ for which u is a stationary.

Observation: Though this method is very simple in its approach, but it fails to determine the nature of the stationary points whether they are points of maximum or minimum or saddle points.

Solution: Let edges of the parallelopiped be $2x$, $2y$, $2z$ parallel to the coordinate axes so that volume,

$$V = 8xyz \qquad \ldots(1)$$

Our object is to maximise $V(x, y, z)$ subject to the condition,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{or} \quad \phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad \ldots(2)$$

Define function,

$$F(x, y, z) = V + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \qquad \ldots(3)$$

So that for stationary values,

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 8yz + \lambda\frac{2x}{a^2} = 0 \qquad \ldots(i) \\ \frac{\partial F}{\partial y} &= 8xz + \lambda\frac{2y}{b^2} = 0 \qquad \ldots(ii) \\ \frac{\partial F}{\partial z} &= 8xy + \lambda\frac{2z}{c^2} = 0 \qquad \ldots(iii) \end{aligned}\right\} \qquad \ldots(4)$$

Equating the values of λ from (i) and (ii), (i) and (iii) we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{and} \quad \frac{x^2}{a^2} = \frac{z^2}{c^2}$$

implying thereby $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} = \dfrac{1}{3}$; $\left(\text{using } \dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}\right)$ in (2) $\qquad \ldots(5)$

or $$x = \frac{a}{\sqrt{3}}, \; y = \frac{b}{\sqrt{3}}, \; z = \frac{c}{\sqrt{3}}$$

When $x = 0$, the parallelopiped mearly becomes a rectangular sheet and in that case the volume, $V = 0$.

Hence V is maximum when $x = \dfrac{a}{\sqrt{3}}, y = \dfrac{b}{\sqrt{3}}, z = \dfrac{c}{\sqrt{3}}$ and $V_{\text{Maxi}} = 8xyz = \dfrac{8abc}{3\sqrt{3}}$.

Example 77: Find the maximum value of the function cos*A* cos*B* cos*C*.

Solution: If A, B, C are the angles of a triangle ABC, then

$$A + B + C = 180° \quad \text{(an implicit relation)} \qquad \ldots(1)$$

For $$f(A, B, C) = \cos A \cos B \cos C, \qquad \ldots(2)$$

define Lagrange's function,

$$F(A, B, C) = f + \lambda\phi = \cos A \cos B \cos C + \lambda(A + B + C - \pi) \qquad \ldots(3)$$

Alternately: By Conventional Method;

$$V = x^2\left(y + \frac{h}{3}\right) = x(xy) + \frac{x^2h}{3} \quad \text{or} \quad xy = \left(\frac{V - \frac{x^2h}{3}}{x}\right) = \left(\frac{V}{x} - \frac{xh}{3}\right) \qquad \ldots(5)$$

$$S = 4xy + x\sqrt{x^2 + 4h^2} = 4\left(\frac{V}{x} - \frac{xh}{3}\right) + x(x^2 + 4h^2)^{\frac{1}{2}} \qquad \ldots(6)$$

Thus by elimination of y, surface area S has been made a function of two variables x and h only.

Now for minimum surface area, $S_x = 0 = S_h$.

Therefore, $$S_x(x,h) = \left(-4V \cdot \frac{1}{x^2} - \frac{4}{3}h\right) + \left(\sqrt{x^2 + 4h^2} + \frac{x^2}{\sqrt{x^2 + 4h^2}}\right) = 0 \qquad \ldots(7)$$

and $$S_h(x,h) = -\frac{4}{3}x + \frac{4xh}{\sqrt{x^2 + 4h^2}} = 0 \qquad \ldots(8)$$

From (8) $\quad -\frac{4}{3}x + \frac{4xh}{\sqrt{x^2 + 4h^2}} = 0 \quad \text{or} \quad (x^2 + 4h^2) = 9h^2 \quad \text{or} \quad x = \sqrt{5}h$

Putting $x = \sqrt{5}h$ into (7),

$$\left(-\frac{4}{x^2}x^2\left(y + \frac{h}{3}\right) - \frac{4}{3}h\right) + \left(3h + \frac{5h^2}{3h}\right) = 0, \quad \left(\text{as } V = x^2\left(y + \frac{h}{3}\right)\right)$$

or $$-4y - \frac{4}{3}h - \frac{4}{3}h + 3h + \frac{5}{3}h = 0 \quad \text{or} \quad -4y - 2h = 0 \quad \Rightarrow \quad y = \frac{h}{2}.$$

ASSIGNMENT 9

1. Find the maximum and minimum values of

(*i*) $xy + \frac{a^3}{x} + \frac{a^3}{y}$ (*ii*) $2(x^2 - y^2) - x^4 + y^4$ **[NIT Kurukshetra, 2005]**

2. Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$

3. Find the dimensions of a rectangle box without top with a given capacity so that the material used is minimum.

4. Determine the maxima of the function given by $u = (x + 1)(y + 1)(z + 1)$ subject to the condition $x^a y^b z^c = k$.

[**Hint:** Take log of both functions]

5. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

6. Given $x + y + z = a$, find the maximum value of $x^m\, y^n\, z^p$. **[KUK, NIT Kurukshetra, 2004]**

7. Find the point on the surface $z^2 = xy + 1$ nearest to the origin.

By Leibnitz's Rule 1,

$$\frac{d}{dm}F(m) = \int_0^\infty \frac{\partial}{\partial m}\left(\frac{e^{-ax}\sin mx}{x}\right)dx = \int_0^\infty \frac{\partial}{\partial m}(\sin mx)\cdot\frac{e^{-ax}}{x}dx$$

$$= \int_0^\infty x(\cos mx)\frac{e^{-ax}}{x}dx = \int_0^\infty e^{-ax}\cos mx\, dx$$

$$\frac{dF}{dm} = \left[\frac{e^{-ax}}{\sqrt{(-a)^2+m^2}}(-a\cos mx + m\sin mx)\right]_0^\infty = \frac{a}{a^2+m^2} \qquad \ldots(2)$$

$$\left(\text{using } \int_0^\infty e^{ax}\cos bx\, dx = \frac{e^{ax}}{(a^2+b^2)}(a\cos bx + b\sin bx)\right)$$

Integrating both sides with respect to x,

$$F(m) = a\int\frac{1}{a^2+m^2}dx = a\left(\frac{1}{a}\tan^{-1}\frac{m}{a}\right) + C = \tan^{-1}\frac{m}{a} + C,\ a > 0 \qquad \ldots(3)$$

When $m = 0$; $\left.\begin{array}{l}\text{from (3)}, \ F(0) = \tan^{-1}0 + C = C \\ \text{from (1)}, \ F(0) = 0\end{array}\right\}$ implying $C = 0$

$$\therefore \qquad \int_0^\infty \frac{e^{-ax}\sin mx}{x}dx = \tan^{-1}\frac{m}{a}$$

Example 84: Prove that $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx = \log(1+a)$, (a > – 1)

Solution: Let $I = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx$ $\qquad \ldots(1)$

so that $$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}(1-e^{-ax})\right)dx = \int_0^\infty \frac{e^{-x}}{x}\cdot e^{-ax}(-x)dx = \int_0^\infty e^{-x}e^{-ax}dx$$

$$\frac{dI}{da} = \int_0^\infty e^{-(1+a)x}dx = \left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty = \frac{1}{(1+a)},\ a > -1$$

On integration,

$$I = \log(1+a) + C \qquad \ldots(2)$$

Now when $a = 0$ then $\left.\begin{array}{l}\text{from (1)}, \ I = \int_0^\infty \frac{e^{-x}}{x}(1-1)dx = 0 \\ \text{from (2)}, \ I = \log(1+0) + C = C\end{array}\right\}$ thereby implying $C = 0$

$$\therefore \qquad I = \log(1+a),\ a > -1.$$

Example 85: Show that $\int_0^{\pi/2}\frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi\left(\sqrt{1+y}-1\right)$ [NIT Kurukshetra, 2002]