About this document. These notes were created for use as primary reading material for the graduate course *Math 205A: Complex Analysis* at UC Davis.

The current 2020 revision (dated June 15, 2021) updates my earlier version of the notes from 2018. With some exceptions, the exposition follows the textbook *Complex Analysis* by E. M. Stein and R. Shakarchi (Princeton University Press, 2003).

The notes are typeset in the Bera Serif font.

Acknowledgements. I am grateful to Christopher Alexander, Jennifer Brown, Brynn Caddel, Keith Conrad, Bo Long, Anthony Nguyen, Jianping Pan, and Brad Velasquez for comments that helped me improve the notes. Figure 5 on page 27 was created by Jennifer Brown and is used with her permission. An anonymous contributor added an index and suggested the Bera Serif font and a few other improvements to the document design.

You too can help me continue to improve these notes by emailing me at `romik@math.ucdavis.edu` with any comments or corrections you have.

Complex Analysis Lecture Notes

Document version: June 15, 2021

Cover figure: a heat map plot of the entire function $z \mapsto z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$.

Created with Mathematica using code by Simon Woods, available at

http://mathematica.stackexchange.com/questions/7275/how-can-i-generate-this-domain-coloring-plot

Lemma 2 (Conformality lemma.). *Assume that* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a* 2×2 *real matrix. The following are equivalent:*

(a) A *preserves orientation (that is,* $\det A > 0$*) and is conformal, that is*

$$\frac{\langle Aw_1, Aw_2 \rangle}{|Aw_1|\,|Aw_2|} = \frac{\langle w_1, w_2 \rangle}{|w_1|\,|w_2|}$$

for all $w_1, w_2 \in \mathbb{R}^2$.

(b) A *takes the form* $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ *for some* $a, b \in \mathbb{R}$ *with* $a^2 + b^2 > 0$.

(c) A *takes the form* $A = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *for some* $r > 0$ *and* $\theta \in \mathbb{R}$. *(That is, geometrically* A *acts by a rotation followed by a scaling.)*

Proof that (a) $\Longrightarrow$ *(b).* Note that both columns of A are nonzero vectors by the assumption that $\det A > 0$. Now applying the conformality assumption with $w_1 = (1,0)^\top$, $w_2 = (0,1)^\top$ yields that $(a,c) \perp (b,d)$, so that $(b,d) = \kappa(-c, a)$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. On the other hand, applying the conformality assumption with $w_1 = (1,1)^\top$ and $w_2 = (1,-1)^\top$ yields that $(a+b, c+d) \perp (a-b, c-d)$, which is easily seen to be equivalent to $a^2 + c^2 = b^2 + d^2$. Together with the previous relation that implies that $\kappa = \pm 1$. So A is of one of the two forms $\begin{pmatrix} a & -c \\ c & a \end{pmatrix}$ or $\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$. Finally the assumption that $\det A > 0$ means it is the first of those two possibilities that must occur. □

Exercise 3. Complete the proof of the lemma above by showing the implications (b) $\Longleftrightarrow$ (c) and that (b) $\Longrightarrow$ (a).

Another curious consequence of the Cauchy-Riemann equations, which gives an alternative geometric picture to that of conformality, is that analyticity implies the orthogonality of the level curves of u and of v. That is, if $f = u + iv$ is analytic then

$$\langle \nabla u, \nabla v \rangle = (u_x, u_y) \perp (v_x, v_y) = u_x v_x + u_y v_y = v_y v_x - v_x v_y = 0.$$

Since ∇u (resp. ∇v) is orthogonal to the level curve $\{u = c\}$ (resp. the level curve $\{v = d\}$, this proves that the level curves $\{u = c\}$, $\{v = d\}$ meet at right angles whenever they intersect.

Yet another important and remarkable consequence of the Cauchy-Riemann equations is that, at least under mild assumptions (which we will see later

As a further reminder, the basic result known as the *fundamental theorem of calculus for line integrals* states that if $F = \nabla u$ then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = u(\gamma(b)) - u(\gamma(a)).$$

Definition 2 (contour integrals and arc length intervals)**.** For a function $f = u + iv$ of a complex variable z and a curve γ, define

$$\begin{aligned}\int_\gamma f(z)\,dz &= \text{``}\int_\gamma (u+iv)(dx+idy)\text{''}\\ &= \left(\int_\gamma u\,dx - v\,dy\right) + i\left(\int_\gamma v\,dx + u\,dy\right)\\ &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \qquad \text{(contour integral)},\\ \int_\gamma f(z)\,|dz| &= \int_\gamma f(z)\,ds = \int_\gamma u\,ds + i\int_\gamma v\,ds \qquad \text{(arc length integral)}.\end{aligned}$$

If γ is a *closed* curve (the two endpoints are the same, i.e., it satisfies $\gamma(a) = \gamma(b)$), we denote the contour integral as $\oint_\gamma f(z)\,dz$, and similarly $\oint f(z)\,|dz|$ for the arc length integral.

A special case of an arc length integral is the length of the curve, defined as the integral of the constant function 1:

$$\mathrm{len}(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt.$$

As mentioned above, our convention of mildly abusing terminology puts on us the burden of having to remeber to check that these definitions do not depend on the parametrization of the curve. Indeed: if $\gamma_1 \sim \gamma_2$ are representatives of the same equivalence class of parametrized curves, that is, $\gamma_2(t) = \gamma_1(I(t))$ for some nicely-behaved function, then using a standard change of variables in single-variable integrals we see that

$$\begin{aligned}\int_{\gamma_2} f(z)\,dz &= \int_c^d f(\gamma_2(t))\gamma_2'(t)dt = \int_c^d f(\gamma_1(I(t)))(\gamma_1\circ I)'(t)\,dt\\ &= \int_c^d f(\gamma_1(I(t)))\gamma_1'(I(t))I'(t)\,dt = \int_a^b f(\gamma_1(\tau))\gamma_1'(\tau)\,d\tau\\ &= \int_{\gamma_1} f(z)\,dz.\end{aligned}$$

To make this precise, write

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} \\ &= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}(w-z_0)^{-n-1}(z-z_0)^n.\end{aligned}$$

This is a power series in $z - z_0$ that, assuming $w \in C_R(z_0)$, converges absolutely for all z such that $|z - z_0| < R$ (that is, for all $z \in D_R(z_0)$). Moreover the convergence is clearly uniform in $w \in C_R(z_0)$. Since infinite summations that are absolutely and uniformly convergent can be interchanged with integration operations, we then get, using the extended version of Cauchy's integral formula, that

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\oint_{C_R(z_0)}\frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\oint_{C_R(z_0)} f(w)\sum_{n=0}^{\infty}(w-z_0)^{-n-1}(z-z_0)^n\,dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_R(z_0)} f(w)(w-z_0)^{n-1}\,dw\right)(z-z_0)^n \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,\end{aligned}$$

which is precisely the expansion we were after. □

Remark 4. In the above proof, if we only knew the simple ($n = 0$) case of Cauchy's integral formula (and in particular didn't know the regularity theorem that follows from the extended case of this formula), we would still conclude from the penultimate expression in the above chain of equalities that $f(z)$ has a power series expansion of the form $\sum_n a_n(z-z_0)^n$, with $a_n = (2\pi i)^{-1}\int_{C_R(z_0)} f(w)(w-z)^{-n-1}$. It would then follow from earlier results we proved that $f(z)$ is differentiable infinitely many times, and that $a_n = f^{(n)}(z_0)/n!$, which would again give the extended version of Cauchy's integral formula.

Theorem 15 (Liouville's theorem)**.** *A bounded entire function is constant.*

Proof. An easy application of the (case $n = 1$ of the) Cauchy inequalities gives upon taking the limit $R \to \infty$ that $f'(z) = 0$ for all z, hence, as we already proved, f must be constant. □

positive if the curve goes in the positive direction around the origin; negative if the curve goes in the negative direction around the origin; or zero if there is no net change in the argument. This number is more properly called the **winding number** of f around $w = 0$ (also sometimes referred to as the **index** of the curve around 0), and denoted

$$\mathrm{Ind}_0(f) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z}\,dz.$$

More generally, one can define the winding number at $z = z_0$ as the number of times a curve γ winds around an arbitrary point z_0, which (it is easy to see) will be given by

$$\mathrm{Ind}_{z_0}(f) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz,$$

assuming that γ does not cross z_0.

Note that winding number is a *topological* concept of planar geometry that can be considered and studied without any reference to complex analysis; indeed, in my opinion that is the correct approach. It is possible, and not especially difficult, to define it in purely topological terms without mentioning contour integrals, and then show that the complex analytic and topological definitions coincide. Try to think what such a definition might look like.

Theorem 26 (Rouché's theorem). *Assume that f, g are holomorphic on a region Ω containing a circle $\gamma = C$ and its interior (or, more generally, a toy contour γ and the region U enclosed by it). If $|f(z)| > |g(z)|$ for all $z \in \gamma$ then f and $f + g$ have the same number of zeros inside the region U.*

Proof. Define $f_t(z) = f(z) + tg(z)$ for $t \in [0,1]$, and note that $f_0 = f$ and $f_1 = f + g$, and that the condition $|f(z)| > |g(z)|$ on γ implies that f_t has no zeros on γ for any $t \in [0,1]$. Denote

$$n_t = \frac{1}{2\pi i}\oint_\gamma \frac{f_t'(z)}{f_t(z)}\,dz,$$

which by the argument principle is the number of "generalized zeros" (zeros or poles, counting multiplicities) of f_t in U. In particular, the function $t \mapsto n_t$ is integer-valued. If we also knew that it was continuous, then it would have to be constant (by the easy exercise: any integer-valued continuous function on an interval $[a,b]$ is constant), so in particular we would get the desired conclusion that $n_1 = n_0$.

open covering of the interval $[0,1]$, so again using the Heine-Borel property, we can extract a finite subcovering. This enables us to find a sequence $0 = s_0 < s_1 < \ldots < s_n = 1$ that we claimed exist at the beginning of the proof, namely where the relation

$$\int_{\gamma_{s_{j-1}}} f(z)\,dz = \int_{\gamma_{s_j}} f(z)\,dz$$

holds for each $j = 1, \ldots, n$ (with s_{j-1} playing the role of s and s_j playing the role of s' in the discussion above). ☐

Theorem 29 (Cauchy's theorem (general version))**.** *If f is holomorphic on a simply-connected region Ω, then for any closed curve in Ω we have*

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. Assume for simplicity that γ is parametrized as a curve on $[0,1]$. Then it can be thought of as the concatenation of two curves γ_1 and $-\gamma_2$, where $\gamma_1 = \gamma_{|[0,1/2]}$ and γ_2 is the "reverse" of the curve $\gamma_{|[1/2,1]}$. Note that γ_1 and γ_2 have the same endpoints. By the invariance property of contour integrals under homotopy proved above, we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0.$$

☐

Corollary 9. *Any holomorphic function on a simply-connected region has a primitive.*

Exercise 10. The proof of Theorem 28 above still involves a minor amount of what I call "dishonesty"; that is, the proof is not actually formally correct as written but contains certain inconsistencies between what the assumptions of the theorem are and what we end up actually using in the body of the proof. Can you identify those inconsistencies? What additional work might be needed to fix these problems? And why do you think the author of these notes, and the authors of the textbook [11], chose to present things in this way rather than treat the subject in a completely rigorous manner devoid of any inaccuracies? (The last question is a very general one about mathematical pedagogy; coming up with a good answer might help to demystify for you a lot of similar decisions that textbook authors and course instructors make in the teaching of advanced material, and make the study of such topics a bit less confusing in the future.)

14 The Euler gamma function

The *Euler gamma function* (often referred to simply as the gamma function) is one of the most important special functions in mathematics. It has applications to many areas, such as combinatorics, number theory, differential equations, probability, and more, and is probably the most ubiquitous transcendental function after the "elementary" transcendental functions (the exponential function, logarithms, trigonometric functions and their inverses) that one learns about in calculus. It is a natural meromorphic function of a complex variable that extends the factorial function to non-integer values. In complex analysis it is particularly important in connection with the theory of the Mellin transform (a version of the Fourier transform associated with the multiplicative group of positive real numbers in the same way that the ordinary Fourier transform is associated with the additive group of the real numbers).

Most textbooks define the gamma function in one way and proceed to prove several other equivalent representations of it. However, the truth is that none of the representations of the gamma function is more fundamental or "natural" than the others. So, it seems more logical to start by simply listing the various formulas and properties associated with it and then proving that the different representations are equivalent and that the claimed properties hold.

Theorem 31 (the Euler gamma function)**.** *There exists a unique function $\Gamma(s)$ of a complex variable s that has the following properties:*

1. *$\Gamma(s)$ is a meromorphic function on $\mathbb{C}$.*
2. ***Connection to factorials:*** *$\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.*
3. ***Important special value:*** *$\Gamma(1/2) = \sqrt{\pi}$.*
4. ***Integral representation:***

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx \qquad (\operatorname{Re} s > 0).$$

5. ***Hybrid series-integral representation:***

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-x} x^{s-1}\,dx \qquad (s \in \mathbb{C}).$$

I leave it as an exercise to check (or read the easy explanation in [11]) that the function it defines is holomorphic in that region.

Next, perform an integration by parts, to get that, again for $\mathrm{Re}(s) > 0$, we have

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s \, dx = -e^{-x} x^s \Big|_{x=0}^{x=\infty} + \int_0^\infty e^{-x} s x^{s-1} \, dx = s\,\Gamma(s),$$

which is the functional equation.

Combining the trivial evaluation $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$ with the functional equation shows by induction that $\Gamma(n+1) = n!$.

The special value $\Gamma(1/2) = \sqrt{\pi}$ follows immediately by a change of variable $x = u^2$ in the integral and an appeal to the standard Gaussian integral $\int_{-\infty}^\infty e^{-u^2}\,du = \sqrt{\pi}$:

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2}\,dx = \int_0^\infty e^{-u^2} 2\,du = \int_{-\infty}^\infty e^{-u^2}\,du = \sqrt{\pi}.$$

The functional equation can now be used to perform an analytic continuation of $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$: for example, we can define

$$\Gamma_1(s) = \frac{\Gamma(s+1)}{s}$$

which is a function that is holomorphic on $\mathrm{Re}(s) > -1$, $s \neq 0$ and coincides with $\gamma(s)$ for $\mathrm{Re}(s) > 0$. By the principle of analytic continuation this provides a unique extension of $\Gamma(s)$ to the region $\mathrm{Re}(s) > -1$. Because of the factor $1/s$ and the fact that $\Gamma(1) = 1$ we also see that $\Gamma_1(s)$ has a simple pole at $s = 0$ with residue 1.

Next, for $\mathrm{Re}(s) > -2$ we define

$$\Gamma_2(s) = \frac{\Gamma_1(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)},$$

a function that is holomorphic on $\mathrm{Re}(s) > -2$, $s \neq 0, -1$, and coincides with $\Gamma_1(s)$ for $\mathrm{Re}(s) > -1$, $s \neq 0$. Again, this provides an analytic continuation of $\Gamma(s)$ to that region. The factors $1/s(s+1)$ show that $\Gamma_2(s)$ has a simple pole at $s = -1$ with residue -1.

Continuing by induction, having defined an analytic continuation $\Gamma_{n-1}(s)$ of $\Gamma(s)$ to the region $\mathrm{Re}(s) > -n+1$, $s \neq 0, -1, -2, \ldots, -n+2$, we now define

$$\Gamma_n(s) = \frac{\Gamma_{n-1}(s+1)}{s} = \ldots = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)}.$$

Proof. Lemma 8 above implies that the infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges to a nonzero limit for any $z \in \Omega$. By repeating the same estimates in the proof of that lemma in the context of z being allowed to range on a compact subset $K \subset \Omega$, one sees that the sequence of partial products $\prod_{k=1}^{n}(1+f_n)$ actually converges uniformly on compacts, so the limiting function is holomorphic. □

Proof that $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$ is an entire function.

$$\sum_{n=1}^{\infty}\left|\left(1+\frac{z}{n}\right)e^{-z/n}-1\right| = \sum_{n=1}^{\infty}\left|\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}+O\left(\frac{z^2}{n^2}\right)\right)-1\right|$$
$$= \sum_{n=1}^{\infty}\left|O\left(\frac{z^2}{n^2}\right)\right| < \infty$$

(where the big-O notation hides a universal constant — the dependence on z is encapsulated in the z^2 factor). In particular, the convergence is uniform on compacts on $\mathbb{C}$. So we are almost in the setting of Lemma 9, except that in order to apply that result, which requires the functions participating in the product to be nonzero, one needs to be a bit more careful and separate out the zeros: for a fixed disc $D_{N+1/2}(0)$ of radius $N+1/2$ around 0, consider only the product starting at $n = N+1$ — those functions are nonzero in the disc so the previous result applies to give a function that's holomorphic and nonzero in $D_N(0)$. Then separately the factors $(1+z/n)$, $n = 1,\dots,N$ contribute simple zeros at $z = -1,\dots,-N$. □

Corollary 11 (the reflection formula)**.** $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}$.

Proof.

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = \Gamma(s)^{-1}(-s)^{-1}\Gamma(-s)^{-1}$$
$$= \frac{-1}{s}\cdot s e^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}\cdot(-s)e^{-\gamma s}\prod_{n=1}^{\infty}\left(1-\frac{s}{n}\right)e^{s/n}$$
$$= s\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right) = s\frac{\sin(\pi s)}{\pi s} = \frac{\sin(\pi s)}{\pi},$$

where we used the product representation $\sin(\pi z) = \pi z\prod_{n=1}^{\infty}(1-z^2/n^2)$ for the sine function derived in a homework problem. □

implies that

$$\liminf_{x\to\infty} \psi(x)/x \geq \lim_{x\to\infty} (1-\epsilon)\left(1 - \frac{\log x}{x^\epsilon}\right) = 1-\epsilon.$$

Again, since $\epsilon \in (0,1)$ was arbitrary, it follows that $\liminf_{x\to\infty} \frac{\psi(x)}{x} = 1$. Combining the two results about the lim inf and lim sup proves that $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, as claimed. □

Lemma 12. *For* $\mathrm{Re}(s) > 1$ *we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Proof. Using the Euler product formula and taking the logarithmic derivative (which is an operation that works as it should when applied to infinite products of holomorphic functions that are uniformly convergent on compact subsets), we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\frac{d}{ds}(1-p^{-s})}{1-p^{-s}} = \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}}$$

$$= \sum_p \log p \,(p^{-s} + p^{-2s} + p^{-3s} + \ldots) = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \log p \cdot p^{-ks}$$

$$= \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

□

Lemma 13. *There is a constant* $C > 0$ *such that* $\psi(x) < Cx$ *for all* $x \geq 1$.

Proof. The idea of the proof is that the binomial coefficient $\binom{2n}{n}$ is not too large on the one hand, but is divisible by many primes (all primes between n and $2n$) on the other hand — hence it follows that there cannot be too many primes, and in particular the weighted prime-counting function $\psi(x)$ can be easily bounded from above using such an argument. Specifically, we have that

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} > \binom{2n}{n} \geq \prod_{n<p\leq 2n} p = \exp\left(\sum_{n<p\leq 2n} \log p\right)$$

$$= \exp\left(\psi(2n) - \psi(n) - \sum_{n<p^k\leq 2n,\ k>1} \log p\right).$$

$$\geq \exp\left(\psi(2n) - \psi(n) - O(\sqrt{n}\log^2 n)\right).$$

(The estimate $O(\sqrt{n}\log^2 n)$ for the sum of $\log p$ for prime powers higher than 1 is easy and is left as an exercise.) Taking the logarithm of both sides, this gives the bound

$$\psi(2n) - \psi(n) \le 2n\log 2 + C_1\sqrt{n}\log n \le C_2 n,$$

valid for all $n \ge 1$ with some constant $C_2 > 0$. It follows that

$$\begin{aligned}\psi(2^m) &= (\psi(2^m) - \psi(2^{m-1})) \\ &\qquad + (\psi(2^{m-1}) - \psi(2^{m-2})) + \ldots + (\psi(2^1) - \psi(2^0)) \\ &\le C_2(2^{m-1} + \ldots + 2^0) \le C_2 2^m,\end{aligned}$$

so the inequality $\psi(x) \le C_2 x$ is satisfied for $x = 2^m$. It is now easy to see that this implies the result also for general x, since for $x = 2^m + \ell$ with $0 \le \ell < 2^m$ we have

$$\psi(x) = \psi(2^m + \ell) \le \psi(2^{m+1}) \le C_2 2^{m+1} \le 2C_2 2^m \le 2C_2 x. \qquad \square$$

Theorem 37 (Newman's tauberian theorem)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be a bounded function that is integrable on compact intervals. Define a function $g(z)$ of a complex variable z by*

$$g(z) = \int_0^\infty f(t)e^{-zt}\,dt$$

(g is known as the Laplace transform *of f.) Clearly $g(z)$ is defined and holomorphic in the open half-plane $\mathrm{Re}(z) > 0$. Assume that $g(z)$ has an analytic continuation to an open region Ω containing the closed half-plane $\mathrm{Re}(z) \ge 0$. Then $\int_0^\infty f(t)\,dt$ exists and is equal to $g(0)$ (the value at $z = 0$ of the analytic continuation of g).*

Proof. Define a truncated version of the integral defining $g(z)$, namely

$$g_T(z) = \int_0^T f(t)e^{-zt}\,dt$$

for $T > 0$, which for any T is an entire function of z. Our goal is to show that $\lim_{T\to\infty} g_T(0) = g(0)$. This can be achieved using a clever application of Cauchy's integral formula. Fix some large $R > 0$ and a small $\delta > 0$ (which depends on R in a way that will be explained shortly), and consider the contour C consisting of the part of the circle $|z| = R$ that lies in the half-plane $\mathrm{Re}(z) \ge -\delta$, together with the straight line segment along the line $\mathrm{Re}(z) = -\delta$ connecting the top and bottom intersection points of this circle with the line (see Fig. 6(a)). Assume that δ is small enough so that $g(z)$ (which extends analytically at least slightly to the right of $\mathrm{Re}(z) = 0$) is holomorphic in an open

Recall that $-\zeta'(s)/\zeta(s)$ has a simple pole at $s = 1$ with residue 1 (because $\zeta(s)$ has a simple pole at $s = 1$; it is useful to remember the more general fact that if a holomorphic function $h(z)$ has a zero of order k at $z = z_0$ then the logarithmic derivative $h'(z)/h(z)$ has a simple pole at $z = z_0$ with residue k). So $-\frac{1}{z+1}\cdot\frac{\zeta'(z+1)}{\zeta(z+1)}$ has a simple pole with residue 1 at $z = 0$, and therefore $-\frac{1}{z+1}\cdot\frac{\zeta'(z+1)}{\zeta(z+1)} - \frac{1}{z}$ has a *removable* singularity at $z = 0$. Thus, the identity $g(z) = -\frac{1}{z+1}\cdot\frac{\zeta'(z+1)}{\zeta(z+1)} - \frac{1}{z}$ shows that $g(z)$ extends analytically to a holomorphic function in the set

$$\{z \in \mathbb{C} : \zeta(z+1) \neq 0\}.$$

By the "toy Riemann Hypothesis" — the theorem we proved according to which $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$, $g(z)$ in particular extends holomorphically to an open set containing the half-plane $\mathrm{Re}(z) \geq 0$. Thus, $f(t)$ satisfies the assumption of Newman's theorem. We conclude from the theorem that the integral

$$\int_0^\infty f(t)\,dt = \int_0^\infty (\psi(e^t)e^{-t} - 1)dt = \int_1^\infty \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x}$$
$$= \int_1^\infty \frac{\psi(x) - x}{x^2}\,dx$$

converges.

Proof of the prime number theorem. We will prove that $\psi(x) \sim x$, which we already showed is equivalent to the prime number theorem. Assume by contradiction that $\limsup_{x\to\infty}\frac{\psi(x)}{x} > 1$ or $\liminf_{x\to\infty}\frac{\psi(x)}{x} < 1$. In the first case, that means there exists a number $\lambda > 1$ such that $\psi(x) \geq \lambda x$ for arbitrarily large x. For such values of x it then follows that

$$\int_x^{\lambda x} \frac{\psi(t) - t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^{\lambda} \frac{\lambda - t}{t^2}\,dt =: A > 0,$$

but this is inconsistent with the fact that the integral $\int_1^\infty (\psi(x) - x)x^{-2}\,dx$ converges.

Similarly, in the event that $\liminf_{x\to\infty}\frac{\psi(x)}{x} < 1$, that means that there exists a $\mu < 1$ such that $\psi(x) \leq \mu x$ for arbitrarily large x, in which case we have that

$$\int_{\lambda x}^{x} \frac{\psi(t) - t}{t^2}\,dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2}\,dt =: B < 0,$$

again giving a contradiction to the convergence of the integral. □

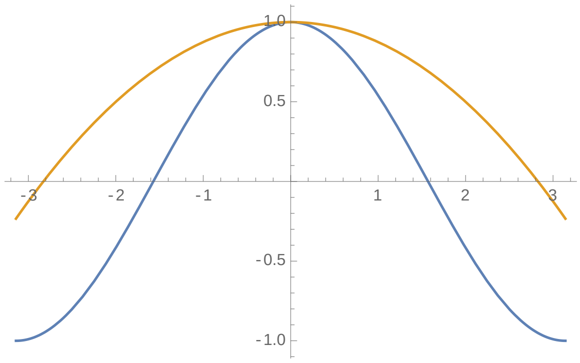


Figure 7: Illustration of the inequality $\cos(t) \leq 1 - t^2/8$.

17.2 Second example: the central binomial coefficient

Let $a_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. A standard way to find the asymptotic behavior for a_n as $n \to \infty$ is to use Stirling's formula. This easily gives that

$$\binom{2n}{n} = (1 + o(1))\frac{4^n}{\sqrt{\pi n}}.$$

(Note that this is not too far from the trivial upper bound $\binom{2n}{n} \leq (1+1)^{2n} = 2^{2n}$.) It is instructive to rederive this result using the saddle-point method, starting from the expansion

$$(1+z)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^n,$$

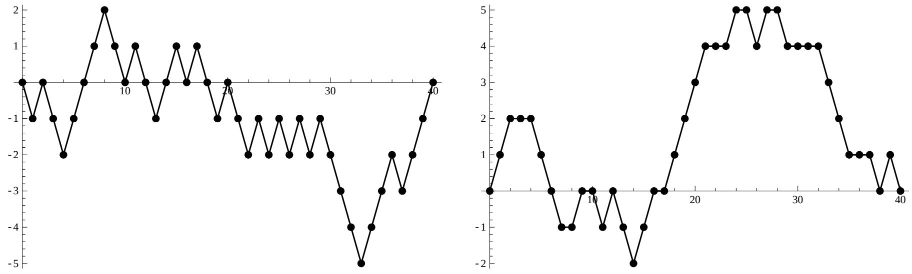
which in particular gives the contour integral representation

$$\binom{2n}{n} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{(1+z)^{2n}}{z^{n+1}} dz.$$

By the same trivial method for deriving upper bounds that we used in the case of the Taylor coefficients $1/n!$ of the function e^z, we have that for each $x > 0$,

$$\binom{2n}{n} \leq (1+x)^{2n}/x^n = \exp\left(\log(1+x) - n\log x\right).$$

We optimize over x by differentiating the expression $\log(1+x) - n\log x$ inside the exponent and setting the derivative equal to 0. This gives $x = 1$, the location of the saddle point. For this value of x, we again recover the trivial inequality $\binom{2n}{n} \leq 2^{2n}$.



(a) (b)

Figure 8: An illustration (with $n = 40$) of the random walks enumerated by (a) the central binomial coefficients and (b) the central trinomial coefficients.

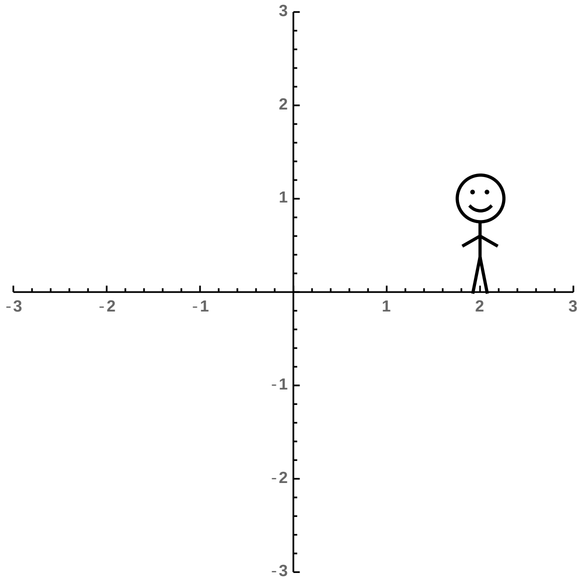
Like their more famous cousins the central binomial coefficients, these coefficients are important in combinatorics and probability theory. Specifically, a_n and b_n correspond to the numbers of random walks on $\mathbb{Z}$ that start and end at 0 and have n steps, where in the case of the central binomial coefficients the allowed steps of the walk are -1 or $+1$, and in the case of the central trinomial coefficients the allowed steps are -1, 0 or 1; see Fig. 8.

Using a saddle point analysis, show that the asymptotic behavior of b_n as $n \to \infty$ is given by

$$b_n \sim \frac{\sqrt{3} \cdot 3^n}{\sqrt{4\pi n}}.$$

17.3 A conceptual explanation

In both the examples of Stirling's formula and the central binomial coefficient we analyzed above, we made what looked like ad hoc choices regarding how to "massage" the integrals, what value r to use for the radius of the contour of integration, what change of variables to make in the integral, etc. Now let us think more conceptually and see if we can generalize these ideas. Note that the quantities we were trying to estimate took a particular form, where for some function $g(z)$ our sequence of numbers could be represented



under each of the following maps $w = f(z)$:

a. $w = \frac{1}{2}z$

b. $w = iz$

c. $w = \overline{z}$

d. $w = (2+i)z - 3$

e. $w = 1/z$

f. $w = z^2 - 1$

6. Prove that the complex numbers a, b, c form the vertices of an equilateral triangle if and only if $a^2 + b^2 + c^2 = ab + ac + bc$.

7. Illustrate the claim from page 11 regarding the orthogonality of the level curves of the real and imaginary parts an analytic functions by drawing (by hand after working out the relevant equations, or using a computer) the level curves of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ for $f = z^2$, $f = e^z$.

8. An immediate corollary of the Fundamental Theorem of Algebra (together with standard properties of polynomials, namely the fact that c is a root of $p(z)$ if and only if $p(z)$ is divisible by the linear factor $z - c$) is that any complex polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$

(where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$), can be factored as

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

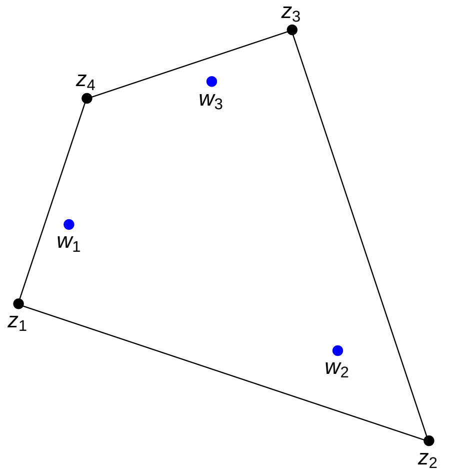


Figure 10: An example of the roots of a complex polynomial and of its derivative. Here $z_1 = 0$, $z_2 = 3 - i$, $z_3 = 2 + 2i$, $z_4 = \frac{1+3i}{2}$ and $w_1 \doteq 0.375 + 0.586i$, $w_2 \doteq 2.336 - 0.335i$, $w_3 \doteq 1.414 + 1.624i$.

(b) Show that assuming a solution to (5) of the form $w = u + v$, the equation (5) for w can be solved by finding a pair u, v of complex numbers such that the equations

$$p = -3uv, \tag{6}$$
$$q = -(u^3 + v^3) \tag{7}$$

are satisfied.

(c) Explain why in order to solve the pair of equations (6)–(7), one can alternatively solve

$$\frac{p^3}{27} = -RS, \tag{8}$$
$$q = -(R + S), \tag{9}$$

where we now denote new unknowns R, S defined by $R = u^3$, $S = v^3$. More precisely, any solution of (6)–(7) can be obtained from *some* (easily determined) solution of (8)–(9).

(d) Explain why the problem of solving (8)–(9) in the unknowns R, S is equivalent to solving the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0 \tag{10}$$

in a (complex) unknown variable t.

27. Let $f(z) = p(z)/q(z)$ be a rational function such that $\deg q \geq \deg p + 2$ (where $\deg p$ denotes the degree of a polynomial p). Prove that the sum of the residues of $f(z)$ over all its poles is equal to 0.

28. (A generalization of the result from problem 21) If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + \ldots + a_0$ is a polynomial of degree n such that for some $0 \leq k \leq n$ we have

$$|a_k| > \sum_{\substack{0 \leq j \leq n \\ j \neq k}} |a_j|,$$

prove that $p(z)$ has exactly k zeros (counting multiplicities) in the unit disk $|z| < 1$.

29. Suggested reading: go to the Mathematics Stack Exchange website (https://math.stackexchange.com) and enter "Rouche" into the search box, to get an amusing list of questions and exercises involving applications of Rouché's theorem to count zeros of polynomials and other analytic functions.

30. Show how Rouché's theorem can be used to give yet another proof of the fundamental theorem of algebra. This proof is one way to make precise the intuitively compelling "topological" proof idea we discussed at the beginning of the course.

31. (a) Draw a simply-connected region $\Omega \subset \mathbb{C}$ such that $0 \notin \Omega$, $1, 2 \in \Omega$, and such that there exists a branch $F(z)$ of the logarithm function on Ω satisfying

$$F(1) = 0, \qquad F(2) = \log 2 + 2\pi i$$

(where $\log 2 = 0.69314\ldots$ is the ordinary logarithm of 2 in the usual sense of real analysis).

(b) More generally, let $k \in \mathbb{Z}$. If we were to replace the above condition $F(2) = \log 2 + 2\pi i$ with the more general condition $F(2) = \log 2 + 2\pi i k$ but keep all the other conditions, would an appropriate simply-connected region $\Omega = \Omega(k)$ exist to make that possible? If so, what would this region look like, roughly, as a function of k?

32. Prove the following properties satisfied by the gamma function:

i. Values at half-integers:

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \qquad (n = 0, 1, 2, \ldots).$$

(e) Show that $\psi(s)$ satisfies the reflection formula

$$\psi(1-s) - \psi(s) = \pi \cot(\pi s).$$

(f)* Here is an amusing application of the digamma function. Consider the sequence of polynomials

$$P_n(x) = x(x-1)\dots(x-n) \qquad (n = 0, 1, 2, \dots)$$

and their derivatives

$$Q_n(x) = P_n'(x).$$

Note that by Rolle's theorem, $Q_n(x)$ has precisely one root in each interval $(k, k+1)$ for $0 \le k \le n-1$. Denote this root by $k + \alpha_{n,k}$, so that the numbers $\alpha_{n,k}$ (the fractional parts of the roots of $Q_n(x)$) are in $(0,1)$.

A curious phenomenon can now be observed by plotting the points $\alpha_{n,k}$, $k = 0, \dots, n-1$ numerically, say for $n = 50$ (Figure 11(a)). It appears that for large n they approximate some smooth limiting curve. This is correct, and in fact the following precise statement can be proved.

Theorem. Let $t \in (0,1)$. Let $k = k(n)$ be a sequence such that $0 \le k(n) \le n-1$, $k(n) \to \infty$ as $n \to \infty$, $n - k(n) \to \infty$ as $n \to \infty$, and $k(n)/n \to t$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \alpha_{n,k(n)} = R(t) := \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi}\log\left(\frac{1-t}{t}\right)\right).$$

In the above formula, $\operatorname{arccot}(\cdot)$ refers to the branch of the inverse cotangent function taking values between 0 and π. The limiting function $R(t)$ is shown in Figure 11(b).

Prove this.

Guidance. Take the logarithmic derivative of $P_n(x)$ to see when the equation $Q_n(x)/P_n(x) = 0$ (which is equivalent to $Q_n(x) = 0$) holds. This will give an equation with a sum of terms. Find a way to separate them into two groups such that the sum in each group can be related, in an asymptotic sense as $n \to \infty$, to the digamma function evaluated at a certain argument (using property (b) above). Take the limit as $n \to \infty$, then simplify using the reflection formula (part (c)).

44. (a) Reprove Theorem 36 (the “toy Riemann hypothesis” — the result that the Riemann zeta function has no zeros on the line $\mathrm{Re}(s) = 1$) by considering the behavior of

$$Y = \mathrm{Re}\left[-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right]$$

for $t \in \mathbb{R} \setminus \{0\}$ fixed and $\sigma \searrow 1$, instead of the quantity

$$X = \log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)|.$$

Use the series expansion

$$-\frac{\zeta(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s},$$

where $\Lambda(n)$ is von Mangoldt’s function (equal to $\log p$ if $n = p^k$ is a prime power, and 0 otherwise).

(b) Try to reprove the same theorem in yet a third way by considering

$$Z = \log|\zeta(\sigma)^{10}\zeta(\sigma+it)^{15}\zeta(\sigma+2it)^6\zeta(\sigma+3it)|$$

and attempting to repeat the argument involving expanding the logarithm in a power series and deducing that $Z \geq 0$. Does this give a proof of the theorem? If not, what goes wrong?

Hint. $(a+b)^6 = a^6 + 6a^5b + 10a^4b^2 + 15a^3b^3 + 10a^2b^4 + 6ab^5 + b^6$.

45. Define arithmetic functions taking an integer argument n, as follows:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1p_2\cdots p_k \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise,} \end{cases}$$
(the Möbius μ-function),

$$d(n) = \sum_{d|n} 1,$$ (the number of divisors function),

$$\sigma(n) = \sum_{d|n} d,$$ (the sum of divisors function),

$$\phi(n) = \#\{1 \leq k \leq n-1 \,:\, \gcd(k,n) = 1\},$$ (the Euler totient function),

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime,} \\ 0 & \text{otherwise,} \end{cases}$$ (the von Mangoldt Λ-function).

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