Solution 2. Let $S(k)$ denote the sum from the problem statement. Then using basic properties of binomial coefficients, one finds that for $k \geq 0$,

$$\begin{aligned} S(k+1) &= \sum_{j=0}^{k+1} 2^{k+1-j} \binom{k+1+j}{j} \\ &= \sum_{j=0}^{k+1} 2^{k+1-j} \left(\binom{k+j}{j} + \binom{k+j}{j-1} \right) \\ &= 2 \sum_{j=0}^{k+1} 2^{k-j} \binom{k+j}{j} + \sum_{j=0}^{k} 2^{k-j} \binom{k+j+1}{j} \\ &= 2\, S(k) + \binom{2k+1}{k+1} + \frac{1}{2} \left(S(k+1) - \binom{2k+2}{k+1} \right) \\ &= 2\, S(k) + \frac{1}{2}\, S(k+1) + \binom{2k+1}{k+1} - \frac{1}{2} \binom{2k+2}{k+1} \\ &= 2\, S(k) + \frac{1}{2}\, S(k+1). \end{aligned}$$

Therefore $S(k+1) = 4\, S(k)$, and since $S(0) = 1$, by induction we have $S(k) = 4^k$ for all k.

Solution 3. Note that the desired sum

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j} = 2^k \sum_{j=0}^{k} 2^{-j} \binom{k+j}{k}$$

is the coefficient of x^k in the polynomial

$$\begin{aligned} P_k(x) &= 2^k \sum_{j=0}^{k} 2^{-j} (1+x)^{k+j} \\ &= 2^k (1+x)^k \sum_{j=0}^{k} \left(\frac{1+x}{2} \right)^j \\ &= 2^k (1+x)^k \, \frac{1 - \left(\frac{1+x}{2}\right)^{k+1}}{1 - \frac{1+x}{2}} \\ &= 2^{k+1} (1+x)^k \, \frac{1 - \left(\frac{1+x}{2}\right)^{k+1}}{1-x} \\ &= \left[2^{k+1}(1+x)^k - (1+x)^{2k+1} \right] \frac{1}{1-x} \\ &= \left[2^{k+1}(1+x)^k - (1+x)^{2k+1} \right] (1 + x + x^2 + \ldots). \end{aligned}$$

But this coefficient can also be expressed as

$$2^{k+1} \sum_{j=0}^{k} \binom{k}{j} - \sum_{j=0}^{k} \binom{2k+1}{j} = 2^{k+1} \cdot 2^k - \frac{1}{2} \cdot 2^{2k+1} = 2^{2k} = 4^k,$$

as claimed.

cases, after making the indicated first move, Alice can use Bob's strategy from the previous paragraph to win.

Comment. The game with k pegs and n holes is equivalent to the game with $n-k$ pegs and n holes (moving the k pegs to the right is equivalent to moving the $n-k$ vacant spaces to the left). This symmetry can be used to reduce the three cases considered in the second paragraph to just two.

B3.

Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Answer. The expected value is $1 + \ln(1/\delta)$.

Solution 1. Let $\rho_n(x)$ be the probability density for the location of x_n. Note that $0 \le x_n \le 1$ for all n, so these density functions all have support $[0, 1]$. They can be found recursively from $\rho_1(x) = 1$ and

$$\rho_{n+1}(x) = \int_{y=x}^{1} \rho_n(y) \frac{dy}{y}.$$

This yields

$$\rho_2(x) = \int_{y=x}^{1} \frac{dy}{y} = -\ln(x); \quad \rho_3(x) = \int_{y=x}^{1} (-\ln y) \frac{dy}{y} = \frac{[-\ln(x)]^2}{2},$$

which suggests that in general

$$\rho_n(x) = \frac{[-\ln(x)]^{n-1}}{(n-1)!};$$

this is straightforward to check by induction.

Let q_n be the probability that $x_n < \delta$ but $x_{n-1} \ge \delta$, that is, the probability that $Z = n$. Then $q_1 = \delta$, and for $n \ge 2$ we have

$$\begin{aligned} q_n &= \int_0^{\delta} \rho_n(x) - \rho_{n-1}(x)\, dx \\ &= \int_0^{\delta} \frac{[-\ln(x)]^{n-1}}{(n-1)!} - \frac{[-\ln(x)]^{n-2}}{(n-2)!}\, dx \\ &= \left. \frac{x[-\ln(x)]^{n-1}}{(n-1)!} \right|_0^{\delta} \\ &= \frac{\delta[-\ln(\delta)]^{n-1}}{(n-1)!}. \end{aligned}$$

Letting $g(t) = f(1/t)$ and making the substitution $u = tx$, this becomes

$$g(t) = 1 + \frac{1}{t}\int_1^t g(u)\,du.$$

Since f is monotone decreasing, g is monotone increasing and hence integrable. Thus it follows from this functional equation that g is continuous for $t > 0$. Hence the integral in the functional equation is a differentiable function of t, and it follows that g is differentiable.

Multiplying both sides of the functional equation by t and then taking the derivative of both sides leads to

$$g(t) + tg'(t) = 1 + g(t), \quad \text{so} \quad tg'(t) = 1.$$

Integrating and using the initial condition $g(1) = 1$, we get $g(t) = 1 + \ln t$ and hence $f(\delta) = 1 + \ln(1/\delta)$.

B4. Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\dfrac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$.

Evaluate $M(2020)$.

Answer. $\dfrac{1}{4040}$.

Solution. We will show that $M(n) = \dfrac{1}{2n}$ for all n, by partitioning V_n into subsets such that the average of $\dfrac{1}{q(\mathbf{v})}$ over each subset is $\dfrac{1}{2n}$. First note that giving an element $\mathbf{v} \in V_n$ is equivalent to giving a sequence of length $2n$ consisting of symbols U (for "up") and D (for "down") so that each symbol occurs n times in the sequence; the symbol in position i is U or D according to whether $s_i - s_{i-1}$ is 1 or -1. With this representation of elements of V_n, there is a natural "cyclic rearrangement" map $\sigma : V_n \to V_n$ which moves each of the symbols one position back cyclically, that is, the symbol in position 1 moves to position $2n$, and for every $j > 1$ the symbol in position j moves to position $j-1$. In terms of the $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$, this works out to

$$\sigma(\mathbf{v}) = (t_0, t_1, \cdots, t_{2n-1}, t_{2n}) \quad \text{where} \quad t_j = s_{j+1} - s_1\,,$$

with the understanding that subscripts are taken modulo $2n$. (Note that $t_0 = t_{2n} = 0$ and that $|t_j - t_{j-1}| = |s_{j+1} - s_j| = 1$.)

From the representation using the symbols U and D, we see that $\sigma^{2n}(\mathbf{v}) = \mathbf{v}$. In particular, for any $\mathbf{v} \in V_n$, the list of elements $\mathbf{v}, \sigma(\mathbf{v}), \sigma^2(\mathbf{v}), \ldots, \sigma^{2n-1}(\mathbf{v})$ runs through the orbit under σ of $\mathbf{v}$ a whole number of times. So the average of $\dfrac{1}{q(\mathbf{w})}$ for $\mathbf{w}$ on that list of elements is the same as the average over the orbit of $\mathbf{v}$; because the orbits partition V_n, it is enough to show that this average is $\dfrac{1}{2n}$ for any $\mathbf{v}$.