13.4 Answers to Selected Exercises

13.2. Let T be the random variable that is the angle between the positron trajectory and the μ^+-spin

$$\mu_2 = E_\alpha T^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2(1+\alpha\cos t)\,dt = \frac{\pi^2}{3} - 2\alpha$$

Thus, $\alpha = (\mu_2 - \pi^2/3)/2$. This leads to the method of moments estimate

$$\hat\alpha = \frac{1}{2}\left(\overline{t^2} - \frac{\pi^2}{3}\right)$$

where $\overline{t^2}$ is the sample mean of the square of the observations.

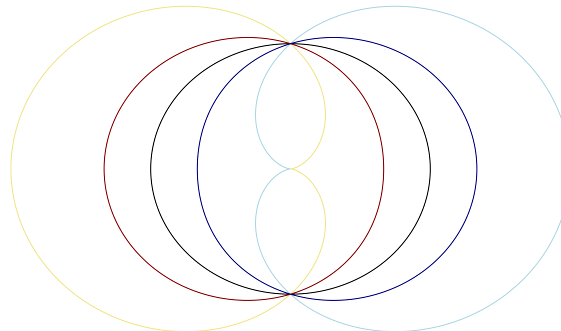

Figure 13.2: Densities $f(t|\alpha)$ for the values of $\alpha = -1$ (yellow). $-1/3$ (red), 0 (black), 1/3 (blue), 1 (light blue).

13.4. Let X be the random variable for the number of tagged fish. Then, X is a hypergeometric random variable with

$$\text{mean } \mu_X = \frac{kt}{N} \quad \text{and variance} \quad \sigma_X^2 = k\frac{t}{N}\frac{N-t}{N}\frac{N-k}{N-1}$$

$$N = g(\mu_X) = \frac{kt}{\mu_X}. \quad \text{Thus, } g'(\mu_X) = -\frac{kt}{\mu_X^2}.$$

The variance of $\hat N$

$$\mathrm{Var}(\hat N) \approx g'(\mu)^2\sigma_X^2 = \left(\frac{kt}{\mu_X^2}\right)^2 k\frac{t}{N}\frac{N-t}{N}\frac{N-k}{N-1} = \left(\frac{kt}{\mu_X^2}\right)^2 k\frac{t}{kt/\mu_X}\frac{kt/\mu_X - t}{kt/\mu_X}\frac{kt/\mu_X - k}{kt/\mu_X - 1}$$

$$= \left(\frac{kt}{\mu_X^2}\right)^2 k\frac{\mu_X t}{kt}\frac{kt-\mu_X t}{kt}\frac{kt-k\mu_X}{kt-\mu_X} = \left(\frac{kt}{\mu_X^2}\right)^2 \mu_X\frac{k-\mu_X}{k}\frac{k(t-\mu_X)}{kt-\mu_X}$$

$$= \frac{k^2t^2}{\mu_X^3}\frac{(k-\mu_X)(t-\mu_X)}{kt-\mu_X}$$

Now if we replace μ_X by its estimate r we obtain

$$\sigma_{\hat N}^2 \approx \frac{k^2t^2}{r^3}\frac{(k-r)(t-r)}{kt-r}.$$

For $t = 200, k = 400$ and $r = 40$, we have the estimate $\sigma_{\hat N} = 268.4$. This compares to the estimate of 276.6 from simulation.

For $t = 1709, k = 6375$ and $r = 138$, we have the estimate $\sigma_{\hat N} = 6373.4$.