**Example**: If $A=\begin{bmatrix}1 & 0 & 2 & 5\\ 2 & -1 & 3 & 7\end{bmatrix}$, then $A'=\begin{bmatrix}1 & 2\\ 0 & -1\\ 2 & 3\\ 5 & 7\end{bmatrix}$

## Properties of Transpose of a Matrix

If $A'$ and $B'$ denote the transposes of A and B respectively, then

(a) $(A')'=\mathrm{A}$ i.e., **the transpose of the transpose of a matrix is the matrix itself**.

**(b)** $(A+B)'=\mathrm{A}'+\mathrm{B}'$ i.e., **the transpose of the sum of two matrices is equal to the sum of their transposes.**

**(c)** $(AB)'=\mathrm{B}'\mathrm{A}'$ i.e., **the transpose of the product of two matrices is equal to the product of their transposes taken in the reverse order.**



## 1.5: Symmetric Matrix

A square matrix $\mathrm{A}=\{\mathrm{a_{ij}}\}$ is said to be symmetric if $\mathrm{A}'=\mathrm{A}$ i.e., if the **transpose of the matrix** is equal to the matrix itself.

Thus, for a symmetric matrix $\mathrm{A}=\{\mathrm{a_{ij}}\}, a_{ij}=a_{ji}$ .

Example : $\begin{bmatrix}-1 & 2 & 3\\ 2 & 5 & 6\\ 3 & 6 & 4\end{bmatrix},\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$ are symmetric matrices .

## 1.6: Skew-Symmetric Matrix

A square matrix $\mathrm{A}=\{\mathrm{a_{ij}}\}$ is said to be **symmetric** if $\mathrm{A}'=-\mathrm{A}$ i.e., if the transpose of the matrix is equal to the negative of the matrix.

Thus, for a symmetric matrix $\mathrm{A}=\{\mathrm{a_{ij}}\}, a_{ij}=-a_{ji}$ .

Putting j=i, $a_{ii}=-a_{ii}\Rightarrow 2a_{ii}=0$ or $a_{ii}=0$ for all $i$. Thus, **all diagonal elements of a skew-symmetric matrix are zero.**

Example : $\begin{bmatrix}0 & 2 & -3\\ -2 & 0 & 1\\ 3 & -1 & 0\end{bmatrix},\begin{bmatrix}0 & h & -g\\ -h & 0 & f\\ g & -f & 0\end{bmatrix}$ are skew-symmetric matrices.

is the conjugate transpose of A.

**Solution:** $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

$$A^{\theta} = \overline{(A')} = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$A^{\theta}A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B(\text{say})$$

Now $B' = \begin{bmatrix} 30 & 6+8i & -19+17i \\ 6-8i & 10 & -5+5i \\ -19-17i & -5+5i & 30 \end{bmatrix}$

$$B^{\theta} = \overline{(B')} = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B$$

Hence $B = A^{\theta}A$ is a Hermitian matrix.

**Example 2:** If A and B are Hermitian, show that AB-BA is skew-Hermitian.

**Solution:** A and B are Hermitian $\Rightarrow A^{\theta} = A$ and $B^{\theta} = B$

Now $(AB - BA)^{\theta} = (AB)^{\theta} - (BA)^{\theta}$

$B^{\theta}A^{\theta} - A^{\theta}B^{\theta} = BA - AB = -(AB - BA)$

$\Rightarrow AB - BA$ is skew-Hermitian.

**Example3:** If A is a skew-Hermitian matrix, then show that iA is Hermitian.

**Solution:** A is a skew- Hermitian matrix $\Rightarrow A^{\theta} = -A$

Now $(iA)^{\theta} = \bar{i}A^{\theta} = (-i)(-A) = iA$

$$A = \begin{bmatrix} 2 & 2-2i & 1-4i \\ 2+2i & 3 & i \\ 1+4i & -i & 9 \end{bmatrix} + \begin{bmatrix} 2i & 2+2i & 4-i \\ -2+2i & i & 4+i \\ -4-i & -4+i & 0 \end{bmatrix}$$

**Example 6:** Show that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

**Solution:** Given $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore A^{\theta} = \overline{(A')} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Now $AA^{\theta} = \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Similarly $A^{\theta}A = I.$

$\because AA^{\theta} = I = A^{\theta}A.$

Hence A is unitary.

**Example 7:** If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ is a matrix, then find $(I-N)(I+N)^{-1}$ and show that it is a unitary matrix.



**Solution:** Given

$$N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

Also

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $I\text{-}N = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$$I + N = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1-\left(-1+4i^2\right) = 6$$

$$\text{Adj}\,(I+N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

## Properties of Inverse of a Matrix

(1) Inverse of A exists only if $|A| \neq 0$ i.e. A is non-singular matrix.

(2) Inverse of a matrix is unique.

(3) Inverse of a `product is the product of inverses in the reverse order

i.e., $(AB)^{-1}=B^{-1}A^{-1}$

(4) Transposition and inverse are commutative i.e.,$(A^{-1})^T = (A^T)^{-1}$, $(A^{-1})^{-1}=A$.

**Example1: Find the inverse of A by Gauss-Jordan method where**

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$$

**Solution :** Writing A=IA i.e.,

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

By $R_{21}(-2)$, $R_{31}(-3)$,we get

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -1\\ 0 & -1 & -3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}A$$

By $R_{23}$ $$\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -3\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -3 & 0 & 1\\ -2 & 1 & 0\end{bmatrix}A$$

By $R_{13}(3)$ ,$R_{23}(-3)$ $$\begin{bmatrix}1 & 2 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}-5 & 3 & 0\\ 3 & -3 & 1\\ -2 & 1 & 0\end{bmatrix}A$$

By $R_2(-1)$ ,$R_3(-1)$ $$\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-5 & 3 & 0\\ -3 & 3 & -1\\ 2 & -1 & 0\end{bmatrix}A$$

(4) Let A be any matrix (square or rectangular).From this matrix A, delete all columns and rows leaving a certain p columns and p rows. Now if p > 1, then the elements which have been left, constitute a square matrix of order p. The determinant of this square matrix is called a **minor of A of order p.**

## 1.13 Nullity of a matrix

Let A be a square matrix of order n and if the rank of A is r, then n-r is called the **nullity of the matrix A** and is usually denoted by N (A).

Thus, Nullity of A i.e. N(A) = Number of column – Rank of A = n- r

**Example 1**: Find the rank and nullity of the following matrices:

$$A=\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 7\\ 3 & 3 & 0\end{bmatrix}$$

**Solution:**

$$A=\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 7\\ 3 & 3 & 0\end{bmatrix}$$

$$\therefore |A| = \begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 7\\ 3 & 3 & 0\end{bmatrix} = -7 \neq 0$$

Rank of A = 3 and Nullity N(A) = 3-3 = 0.

**Example 2**: Find the rank and nullity of the following matrices:

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 2 & 5 & 8\end{bmatrix}$$

**Solution:** $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 2 & 5 & 8\end{bmatrix}$

## Determine the rank of the following matrices by reducing to Echelon form:

**Example 1 : Determine the rank of the following matrices by reducing to Echelon form:**

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$$

Solution: Apply elementary row operations on A

$$\text{By } R_{21}(-2),\ R_{31}(1/2) \quad \sim \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero row is one. So the rank of A is one.

**Example 2**: $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find rank of A, rank of B, rank of A+B, rank of AB and rank of BA.



Solution:
$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix}$$

$$\text{By } R_{31}(-2) \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\text{By } R_{32}(-1) \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of A is 2 since the number of non-zero rows is 2.

$$C_{32}(1), C_{42}(-2)\text{post}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Thus $I_2$=PAQ where

$$P=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}, Q=\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}, \text{Rank of } A=2.$$

**Example 2**: Find the non-singular matrices P and Q such that the normal form of A is PAQ where

$$A=\begin{bmatrix}1&-1&-1\\1&1&1\\3&1&1\end{bmatrix}$$ .Hence find the rank

**Solution:** Consider $A_{3X3} = I_{3X3}\, A_{3X3}\, I_{3X3}$



$$\begin{bmatrix}1&-1&-1\\1&1&1\\3&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$C_{21}(1), C_{31}(1), \text{post}\begin{bmatrix}1&0&0\\1&2&2\\3&4&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$$

$$R_{21}(-1), R_{31}(-3), \text{pre}\begin{bmatrix}1&0&0\\0&2&2\\0&4&4\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-3&0&1\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$$

$$R_2\left(\frac{1}{2}\right), R_3\left(\frac{1}{4}\right), \text{pre}\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-\frac{1}{2}&\frac{1}{2}&0\\\frac{-3}{4}&0&\frac{1}{4}\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$$

**Example 4**: Prove that the following equations are consistent and solve
$2x+4y-z=9, 3x-y+5z=5, 8x+2y+9z=19.$

**Solution:** The matrix equation AX=B is given by

$$\begin{bmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 8 & 2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 5 \\ 19 \end{bmatrix} \qquad \textbf{(1)}$$

$$\therefore [A:B]=\begin{bmatrix} 2 & 4 & -1:9 \\ 3 & -1 & 5:5 \\ 8 & 2 & 9:19 \end{bmatrix}$$

$$R_{21}(-1), R_{31}(-4) \sim \begin{bmatrix} 2 & 4 & -1:9 \\ 1 & -5 & 6:-4 \\ 0 & -14 & 13:-17 \end{bmatrix}$$

$$R_{12} \sim \begin{bmatrix} 1 & -5 & 6:-4 \\ 2 & 0 & -1:9 \\ 0 & -14 & 13:-17 \end{bmatrix}$$

$$R_{21}(-2) \sim \begin{bmatrix} 1 & -5 & 6:-4 \\ 0 & 14 & -13:17 \\ 0 & -14 & 13:-17 \end{bmatrix}$$

$$R_{32}(1) \sim \begin{bmatrix} 1 & -5 & 6:-4 \\ 0 & 14 & -13:17 \\ 0 & 0 & 0:0 \end{bmatrix}$$

$$R_2\left(\tfrac{1}{14}\right) \sim \begin{bmatrix} 1 & -5 & 6:-4 \\ 0 & 1 & -\tfrac{13}{14}:\tfrac{17}{14} \\ 0 & 0 & 0:0 \end{bmatrix}$$

which is in Echelon form.∴ Rank of [A:B] =2=Rank of A < Number of unknowns.

∴ The given equations are consistent and have infinite many solutions. The equations (1) becomes

$$\begin{bmatrix} 1 & -5 & 6 \\ 0 & 1 & -\tfrac{13}{14} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -4 \\ \tfrac{17}{14} \\ 0 \end{bmatrix}$$

$$\therefore\ x-5y+6z=-4,\ y-\left(\tfrac{13}{14}\right)z=\tfrac{17}{14}.$$

Taking z = k (arbitrary value) $x=(-19k+29)/14,\ y=(13k+17)/14,\ z=k.$

**Example 5**:

Prove that what values of $\lambda, \mu$ the equations

$$x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu$$

Have (i) no solution (ii) a unique solution (iii) infinite many solutions.

Solution: The matrix equation AX=B is given by

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix} \qquad (1)$$

$$\therefore [\mathrm{A}:\mathrm{B}]=\begin{bmatrix}1 & 1 & 1:6\\ 1 & 2 & 3:10\\ 1 & 2 & \lambda:\mu\end{bmatrix}$$

$$\mathrm{R}_{21}(-1), R_{32}(-1) \sim \begin{bmatrix}1 & 1 & 1 & :6\\ 0 & 1 & 2 & :4\\ 0 & 0 & \lambda-3: & \mu-10\end{bmatrix}$$



Now consider the following cases:

**Case I.** If $\lambda \neq 3$ ,then Rank of A = Rank of [A:B]=3=Number of unknowns.

Hence in this case the equations are consistent and will have a unique solution.

**Case II.** If $\lambda=3, \mu=10$ ,then Rank of A = Rank of [A:B]= 2 < 3 (Number of unknowns).

Hence in this case the equations are consistent and will have infinite many solutions.

**Case III.** If $\lambda=3, \mu\neq 10$ ,then Rank of A = 2,Rank of [A:B]= 3. Therefore ,Rank of A≠ Rank of [A:B].

Hence in this case the equations are inconsistent and have no solution.

**Example 6**:

Prove that what values of $\lambda, \mu$ the equations

$$x+y+z=1,\ x+2y+4z=\lambda,\ x+4y+10z=\lambda^2$$

**Solution:** The given system of equations is homogeneous and hence it is consistent.

Here $AX = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0.$ (1)

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix}$$

$$R_{21}(-2), R_{31}(3) \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 21 & 0 & -7 \end{bmatrix}$$

$$R_2\left(-\frac{1}{4}\right), R_3\left(\frac{1}{7}\right) \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$



$\therefore$ Rank of A=2 < Number of unknowns,(n=4)

The system of equations is consistent and have infinite many solutions.

Here n-rank of A = 4-2=2.

Hence arbitrary values will be given to two unknowns.

Now equation (1) becomes $AX = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0.$

$$\Rightarrow 2w + 3x - y - z = 0, 3x - z = 0$$

$$\Rightarrow \text{If } y = k_1, z = k_2, \text{then } x = \frac{1}{3}k_2, w = \frac{1}{2}k_1.$$

**Example 9**: Show that the equations :

$$-2x + y + z = a, x - 2y + z = b, x + y - 2z = c.$$

Then geometrically each vector on the line through the origin determined by X gets mapped back onto the same line under multiplication by A. The algebraic eigen value problem consists of determination of such vectors X, known as eigen vectors, such scalars $\lambda$ , known as eigen values. Thus the finding of non-zero vectors that get mapped into scalar multiples of themselves under a linear operator are most important in the study of vibrations of beams, probability (Markov process), Economics (Leontief model),genetics, quantum mechanics, population dynamics and geometry. For example in a mechanical system, they represent the normal modes of vibration.

## Eigen values

If A is a square matrix of order n, we can form the matrix $A - \lambda I$. where $\lambda$

is a scalar and I is the unit matrix of order n.The determinant of this matrix equated to zero,i.e.,

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the **characteristic equation** of A.

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in

$$\lambda \text{ of the form } (-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots\ldots + k_n = 0$$

The roots of this equation are called **characteristic roots or latent roots or eigenvalues of A**.

## Eigen vectors

Consider the linear transformation Y= AX **(1)**

that transforms the column vector X into the column vector Y. In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into $\lambda X$ ($\lambda$ being a non - zero scalar)

According to the transformation (1).

Then $Y = \lambda X$ (2)

From (1) and (2), $AX = \lambda X \Rightarrow AX - \lambda IX = 0 \Rightarrow (A - \lambda I)X = 0$ (3)

This matrix equation gives n homogeneous linear equations

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad (4)$$

These equations will have a non-trivial solution only if the co-efficient matrix (5)

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0 \Rightarrow \lambda = 2,3,5$$

The eigen vector corresponding to eigen value $\lambda = 2$ is given by

$(A - \lambda I)X_1 = 0 \Rightarrow (A - 2I)X_1 = 0$

$$\Rightarrow \begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_{23}(-2) \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_{23} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_1 + x_2 + 4x_3 = 0, 3x_3 = 0$ or $x_3 = 0. \therefore x_2 = -x_1$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ is the eigen vector for } \lambda = 2.$$



The eigen vector corresponding to eigen value $\lambda = 3$ is given by

$(A - \lambda I)X_2 = 0 \Rightarrow (A - 3I)X_2 = 0$

$$\Rightarrow \begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_2 + 4x_3 = 0, -x_2 + 6x_3 = 0, 2x_3 = 0.$

Solving these equations, we have

$x_3 = 0, x_2 = 0, x_1 = 1(\text{say})$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ is the eigen vector for } \lambda = 3.$$

The eigen vector corresponding to eigen value $\lambda = 5$ is given by

$(A - \lambda I)X_3 = 0 \Rightarrow (A - 5I)X_3 = 0$

$$\Rightarrow \begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

On simplification, $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Now, to verify Cayley-Hamilton theorem we have to show that

$$A^3 - 5A^2 + 7A - 3I = O \qquad (1)$$

For $A^{-1}$, Pre multiplying equation 1) by $A^{-1}$

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1}O$$

$$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = O$$

$$\Rightarrow 3A^{-1} = A^2 - 5A + 7I \qquad (2)$$

Now, $A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

From equation (2), $3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A$+I

$= A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 8A^2 - 2A$+I

$= A^5(O) + A^4 - 5A^3 + 8A^2 - 2A$+I .using equation (1)

$= A^4 - 5A^3 + 8A^2 - 2A$+I

$= A^4 - 5A^3 + 7A^2 - 3A + A^2 + A$+I

$= A(A^3 - 5A^2 + 7A - 3I)$+$A^2 + A$+I

$= A(O)$+$A^2 + A$+I , using equation (*1)

=$A^2 + A$+I

The modal matrix of A is $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

And $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Now we have to find $M^{-1}$:

$$M \sim I$$

$$\begin{bmatrix} \mathbf{1} & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , R_3 \to R_3 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & \mathbf{1} & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad , R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad , R_3 \to \frac{R_3}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} , R_1 \to R_1 - R_3 \, , R_2 \to R_2 - R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \sim \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

We know that, $M^{-1}AM = D$

Pre multiplying equation (α) by $M$ and post multiplying by $M^{-1}$

$$M(M^{-1}AM)M^{-1} = MDM^{-1}$$