Damping and Driving Forces

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1 Air Resistance

1.1 Linear Air Resistance

In freefall, we know there are two forces acting on an object. The gravitational force that accelerates the object to the ground, given by mg, and a force of drag that slows the object down, given by $-bv$. Notice how the force of drag is proportional to the negative of speed, meaning the faster the object travels, the more the force air will exert on it to slow it down. Now lets add together the forces and use Newton's second law. We will take the downwards direction as positive here.

$$mg - bv = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$F = ma$ is written in a slightly different form. Recall that $a = \frac{\mathrm{d}v}{\mathrm{d}t}$. Keep in mind that other than v and t, everything else are constants. Dividing through by m gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{b}{m}v \tag{1}$$

Now, we can multiply both sides by $\frac{dt}{g-\frac{b}{m}v}$ and get

$$\frac{dv}{g - \frac{b}{m}v} = dt \tag{2}$$

Now we just need to integrate both sides to get an expression that relates v and t. The left hand side is

$$\int \frac{dv}{g - \frac{b}{m}v} = \frac{m}{b}\int \frac{dv}{\frac{mg}{b} - v} \tag{3}$$

$$= -\frac{m}{b}\ln|\frac{mg}{b} - v| + C \tag{4}$$

Which is just a simple u-sub. The right hand side is

$$\int dt = t + C \tag{5}$$

We now equate the two expressions. Note that the two constants of integration combine into one big C. After equating them, we rewrite it so that v is in terms of t

$$t = -\frac{m}{b}\ln|\frac{mg}{b} - v| + C \implies v = \frac{mg}{b} - Ae^{-\frac{b}{m}t} \tag{6}$$

Where $A = e^{-\frac{b}{m}C}$. Lets determine A. We know that $v(0) = 0$ since the objects are dropped. Plugging 0 into $v(t)$, we have

$$v(0) = \frac{mg}{b} - A = 0 \implies A = \frac{mg}{b} \tag{7}$$

That means the velocity can be modelled by the function

$$v(t) = \frac{mg}{b}(1 - e^{-\frac{b}{m}t}) \tag{8}$$

To know the terminal speed (the speed at which the object will stop accelerating in either direction), we simply need to evaluate the limit as $t \to \infty$, or

$$\lim_{t \to \infty} \frac{mg}{b}(1 - e^{-\frac{b}{m}t}) = \frac{mg}{b} \tag{9}$$

Since the exponential vanishes at infinity, leaving behind the linear term. If we want to find the object's position with respect to time, we simply need to integrate (8) once.

1.2 Quadratic Air Resistance

When an object is large and travels fast through air, linear air resistance fails to accurately predict its equations of motion. Instead, quadratic air resistance is needed. It is given by $F = -bv^2$ This results in the differential equation

$$mg - bv^2 = m\dot{v}$$

We can repeat the steps above, separating variables and obtaining the full equations of motion.

2 Oscillations

2.1 Resistive Forces

Lets consider the same air resistance, but encountered by the mass on spring system. Using Newton's Laws, we arrive at

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{10}$$

This time, it is advantaguous to solve for the displacement x from the equillibrium point directly rather than solve for the velocity first. We first make an ansatz that

$$x = Ae^{\alpha t} \tag{11}$$

Where A and α are arbitrary coefficients. We now take its derivative and obtain

$$\dot{x} = A\alpha e^{\alpha t} \tag{12}$$
$$\ddot{x} = A\alpha^2 e^{\alpha t} \tag{13}$$

We use these results to substitute into (10), leaving us with the simple task of finding the coefficients A and α

$$Ae^{\alpha t}(m\alpha^2 + b\alpha + k) = 0 \tag{14}$$

We arrive at the quadratic equation

$$m\alpha^2 + b\alpha + k = 0 \tag{15}$$

Which has the solution

$$\alpha = -\frac{b}{2m} \pm \sqrt{\gamma^2 - \omega^2} \tag{16}$$

Where we defined the parameter $\omega_0 \equiv \sqrt{\frac{k}{m}}$ and $\gamma \equiv \frac{b}{2m}$. From here, lets define the quantities

$$\Omega \equiv \sqrt{\gamma^2 - \omega_0^2} \tag{17}$$

$$\eta \equiv \frac{b}{2m} \tag{18}$$

We can now rewrite (16) as

$$\alpha = -\eta \pm \omega \tag{19}$$

And we can begin to examine the results. If $\gamma^2 > \omega_0^2$, then the equation of motion will be

$$x = e^{-\eta t}(Ae^{\omega_0 t} + Be^{-\omega_0 t}) \tag{20}$$

Where A and B are constants determined by the initial conditions. This is known as overdamping, since the particle simply slows down exponentially. If $\gamma^2 < \eta^2$, Ω will be imaginary. If we define $\Omega' \equiv \sqrt{\omega_0^2 - \gamma^2}$ and apply Euler's formula,

$$x = Ae^{-\eta t}\cos(\Omega' t - \phi) \tag{21}$$

Where A and ϕ are constants determined by the initial conditions. This is known as underdamping, since the particle oscillates for a while before settling down.

2.2 Driving Forces

Lets consider the previous system, but this time there exists an external driving force on the particle. This external force can be modelled by the function $F = F_0 \cos\omega_1 t$. Using Newton's Laws, we arrive at the solution that

$$m\ddot{x} + b\ddot{x} + kx = F_0 \cos\omega_1 t \tag{22}$$

If we define the quantities $\omega_0 \equiv \sqrt{\frac{k}{m}}$, $\gamma \equiv \frac{b}{m}$ and $F_1 \equiv \frac{F_0}{m}$. We can rewrite (22) as

$$\ddot{x} + \gamma\ddot{x} + \omega_0^2 x = \frac{F_1}{2}(e^{i\omega_1 t} - e^{-i\omega_1 t}) \tag{23}$$

We know the homogenous solution x_0 from last section, which is simply (21). Assuming underdamping gives us

$$x_0 = Ae^{-\eta t}\cos(\Omega' t - \phi) \tag{24}$$

Now we seek a particular solution. We guess a solution in the form of

$$x_p = B(e^{i\zeta t} - e^{-i\zeta t}) \implies \dot{x}_p = i\zeta B(e^{i\zeta t} + e^{-i\zeta t}) \tag{25}$$

$$\ddot{x}_p = -\zeta^2 B(e^{i\zeta t} - e^{-i\zeta t}) \tag{26}$$

Now we substitute this into (23) and find the coefficients. Immediately, we notice that

$$\zeta = \omega_1 \tag{27}$$

$$B = \frac{F}{2(\omega_0^2 - \omega_1^2 + i\gamma\omega_1)} \tag{28}$$

Now we find that our particular solution x_p becomes

$$x_p = \frac{F}{2(\omega_0^2 - \omega_1^2 + i\gamma\omega_1)}(e^{i\omega_1 t} - e^{-i\omega_1 t}) \tag{29}$$

If we define the quantity $R \equiv \sqrt{\omega_0^2 - \omega_1^2 + (2\gamma\omega_0)^2}$ and simplify (29), we arrive at

$$x_p = \frac{F}{R}(\frac{\omega_0^2 - \omega_1^2}{R}\cos\omega_1 t + \frac{2\omega_1}{R}\sin\omega_1 t) \tag{30}$$

A little rearranging and particular choice of coefficients ϕ_2 and ν gives

$$x_p = \frac{F}{R}\cos(\nu t - \phi_2) \tag{31}$$

Putting everything together we finally have

$$x = Ae^{-\eta t}\cos(\Omega' t - \phi) + \frac{F}{R}\cos(\nu t - \phi_2) \tag{32}$$

Notice how the amplitude of oscillation is proportional to R^{-1}, and R gets smaller when the difference between the natural frequency of the object, ω_0 and the driving frequency ω_1 gets smaller. This means that as the difference between the frequencies gets smaller, the amplitude of oscillation will get larger, and the amplitude is at its maximum when the driving frequency is exactly equal to the natural frequency. This is phenomenon known as resonance. Additionally, notice how the first term will eventually vanish, leaving the motion completely dictated by the driving force. We call the first term the "transient" and the second term the "attractor", since the system gets "attracted" towards the second term as time goes on.

2.3 Two Coupled Oscillators

Imagine you have four identical masses m attached to each other by identical springs with spring constant k, where the outermost masses are pinned down and cannot move, while the middle two masses are confined to move along the x-axis. We denote the positions of each mass, from left to right, as $x_0,\ x_1,\ x_2,\ x_3$. The displacement of the spring between any two masses can be found by the difference of their positions. Using Newton's laws, we can easily find an expression for the positions of the middle two springs.

$$m\ddot{x}_1 = k(x_1 - x_0) - k(x_2 - x_1) \tag{33}$$

$$m\ddot{x}_1 = k(x_2 - x_1) - k(x_3 - x_2) \tag{34}$$

The first terms are positive because of Newton's third law. For example, the first term of (34) is positive because it forms a third law pair with the latter term of (33), and is therefore the reaction force from the spring between the two middle masses. We know that $x_0 = x_3 = 0$, so using this fact and simplifying (33) and (34), we arrive at

$$\ddot{x}_1 = -\omega^2(2x_1 - x_2) \tag{35}$$

$$\ddot{x}_2 = -\omega^2(2x_2 - x_1) \tag{36}$$

Where $\omega \equiv \sqrt{\frac{k}{m}}$. We can now begin to solve. Lets guess a solution in the form

$$x_n = A_n e^{\eta_n t} \implies \ddot{x}_n = A_n \eta^2 e^{i\eta t} \tag{37}$$

Substituting this into (35) and (36), we have

$$-A_1\eta^2 e^{i\eta t} = -\omega^2(2A_1 e^{i\eta t} - A_2 e^{i\eta t}) \tag{38}$$

$$-A_2\eta^2 e^{i\eta t} = -\omega^2(2A_2 e^{i\eta t} - A_1 e^{i\eta t}) \tag{39}$$

We seek to find the coefficients A_n and η. First, we rewrite the above into

$$-A_1\eta^2 = -2\omega^2 A_1 + \omega^2 A_2 \tag{40}$$

$$-A_2\eta^2 = \omega^2 A_1 - 2\omega^2 A_2 \tag{41}$$

Notice that this system of equations can be rewritten in matrix form, giving the eigenvalue equation for η^2 and the eigenvector equation for the vector $< A_1, A_2 >$. Rewriting in vector form gives

$$\begin{bmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = -\eta^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{42}$$

To find η, we simply have to take the determinant of the matrix and set it to 0

$$\begin{vmatrix} -2\omega^2 + \eta^2 & \omega^2 \\ \omega^2 & -2\omega^2 + \eta^2 \end{vmatrix} = 0 \tag{43}$$

$$\eta^2 = \pm\omega^2 + 2\omega^2 \tag{44}$$

$$\eta = \{\pm\omega, \pm\sqrt{3}\omega\} \tag{45}$$

To find A_1 and A_2 is trivial, for $\eta^2 = \omega^2$, $A_1 = A_2 = 1$. For $\eta^2 = 3\omega^2$, $A = -A_2 = 1$. Using the identity that $A(e^{i\omega\theta} + e^{-i\omega\theta}) = \frac{A}{2}\cos(\omega\theta + \phi)$, the equations of motion are

$$x_1 = B_1 \cos(\omega t + \phi_1) + B_2 \cos\left(\sqrt{3}\omega t + \phi_2\right) \tag{46}$$

$$x_2 = B_1 \cos(\omega t + \phi_1) - B_2 \cos\left(\sqrt{3}\omega t + \phi_2\right) \tag{47}$$

Where $B_n = \frac{A_n}{2}$. Whats called the normal modes of oscillation is obtained by setting either of the B_n to 0, and represents what would happen if the intial displacements of the masses are the same.

2.4 Generalised Coupled Oscillators

Now imagine N different masses oscillating in the same manner, with the leftmost mass x_0 and the rightmost mass x_{N+1} fixed. Generally, the equation of motion on any individual mass will be

$$\ddot{x}_n = -\omega^2(2x_n - x_{n+1} - x_{n-1}) \tag{48}$$

If we make the same assumptions as (37) and follow the same steps, we arrive at the eigen-equation

$$\begin{bmatrix} -\omega^2 & 2\omega^2 & -\omega^2 & 0 & \dots & 0 \\ 0 & -\omega^2 & 2\omega^2 & -\omega^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -\omega^2 & 2\omega^2 & -\omega^2 & 0 \\ 0 & \dots & 0 & -\omega^2 & 2\omega^2 & -\omega^2 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_N \\ A_{N+1} \end{bmatrix} = -\eta^2 \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_N \\ A_{N+1} \end{bmatrix} \tag{49}$$

Where we know $A_0 = A_{N+1} = 0$. If we know N, we can then solve for the eigenvectors and eigenvalues to obtain the equations of motion and thus, the normal modes.