Q4. **(12 points)** Given that $y_1(x) = x+1$ is a solution of the differential equation

$$(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0,$$

find a second solution $y_2(x)$ of the equation.

Solution:

The Standard form of the equation is

$\boxed{2}$ $$y'' + \frac{2(1+x)}{1-2x-x^2}y' - \frac{2}{1-2x-x^2}y = 0,$$

$\boxed{1}$ $$P(x) = \frac{2(1+x)}{1-2x-x^2}.$$

The Second Solution is

$\boxed{3}$ $$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}dx$$

$$= (x+1)\int \frac{e^{-\int \frac{2(1+x)}{1-2x-x^2}dx}}{(x+1)^2}dx$$

$\boxed{2}$ $$= (x+1)\int \frac{e^{\ln(x^2+2x-1)}}{(x+1)^2}dx$$

$\boxed{1}$ $$= (x+1)\int \frac{x^2+2x-1}{(x+1)^2}dx$$

$\boxed{1}$ $$= (x+1)\int \left[1 - \frac{2}{(x+1)^2}\right]dx$$

$$= (x+1)\left[x + \frac{2}{x+1}\right]$$

$\boxed{2}$ $$= x^2 + x + 2$$