

Limits

Definitions

Precise Definition : We say $\lim_{x \rightarrow a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

“Working” Definition : We say $\lim_{x \rightarrow a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x \rightarrow a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x \rightarrow a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x \rightarrow \infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x \rightarrow -\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x \rightarrow a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x \rightarrow a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one sided limits

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \quad \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \Rightarrow \lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a} f(x) \neq \lim_{x \rightarrow a^+} f(x) \Rightarrow \lim_{x \rightarrow a} f(x) \text{ Does Not Exist}$$

Properties

Assume $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist and c is any number then,

1. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
2. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
3. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$
5. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$
6. $\lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

Basic Limit Evaluations at $\pm \infty$

Note : $\text{sgn}(a) = 1$ if $a > 0$ and $\text{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x \rightarrow \infty} e^x = \infty$ & $\lim_{x \rightarrow -\infty} e^x = 0$
2. $\lim_{x \rightarrow \infty} \ln(x) = \infty$ & $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x \rightarrow \infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x \rightarrow -\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x \rightarrow \pm \infty} x^n = \infty$
6. n odd : $\lim_{x \rightarrow \infty} x^n = \infty$ & $\lim_{x \rightarrow -\infty} x^n = -\infty$
7. n even : $\lim_{x \rightarrow \pm \infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
8. n odd : $\lim_{x \rightarrow \infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
9. n odd : $\lim_{x \rightarrow -\infty} ax^n + \dots + bx + c = -\text{sgn}(a)\infty$