§4.3 Indeterminate Forms ①

Recall from learning limits of the form:
$\lim_{x\to a} f(x)$,
we did the following:

1. Direct Substitution — Ex) $\lim_{x\to 2} 3x-1 = 3(2)-1 = 5$

2. If we have $\frac{0}{0}$ we factor & try a direct subst. again.
Ex) $\lim_{x\to 5} \frac{3x-15}{x^2-25}$ $\quad \frac{0}{0}$
$= \lim_{x\to 5} \frac{3\cancel{(x-5)}}{\cancel{(x-5)}(x+5)} = \frac{3}{5+5} = \frac{3}{10}$

3. If we have a nonzero number over 0, we use one-sided limits to solve.
Ex) $\lim_{x\to 0^+} \frac{1}{x}$ $\quad \frac{1}{0}$
$= \infty$

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There are 7 types of indeterminate forms: ②
$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$,
0^0, ∞^0, 1^∞

L'Hopital's Rule:
Suppose functions $f(x)$ & $g(x)$ are differentiable on (a,b) & $g'(x)\neq 0$ and $a<c<b$, and either
$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\infty}$
then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

Ex) $\lim_{x\to 0}\frac{1-\cos(ax)}{1-\cos(bx)}$, $a,b\in\mathbb{R}$ $\quad \frac{1-1}{1-1} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{a\cdot\sin(ax)}{b\cdot\sin(bx)} = \frac{a\cdot\sin(0)}{b\cdot\sin(0)} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{a^2\cos(ax)}{b^2\cos(bx)} = \frac{a^2\cos(0)}{b^2\cos(0)} = \frac{a^2}{b^2}$.

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Ex) $\lim_{x\to 0}\frac{10^x-e^x}{x}$ $\quad \frac{10^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ ③

$\overset{L'H}{=} \lim_{x\to 0}\frac{10^x\cdot\ln(10)\cdot 1 - e^x\cdot\cancel{\ln e}^{1}\cdot 1}{1}$

$= \lim_{x\to 0}\frac{10^x\cdot\ln(10) - e^x}{1} = 10^0\cdot\ln(10) - e^0$
$= \boxed{\ln(10)-1}$

Note: If $\lim_{x\to c}\frac{f(x)}{g(x)}$ is not an indeterminate form then L'Hopital's Rule cannot be applied.

Ex) $\lim_{x\to 1}\frac{x^2+1}{x^2+2} = \frac{1+1}{1+2} = \frac{2}{3}$.

If we apply L'H:
$\lim_{x\to 1}\frac{2x}{2x} = 1$ } Wrong answer.

Sometimes when we evaluate, we get the form $0\cdot\infty$. In this case we need to rewrite the expression so that it has $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form before we use L'H.

Ex) $\lim_{x\to 0^+} x\ln\left(\frac{1}{x}\right)$ $\quad 0\cdot\infty$, so we need to rewrite this.

$= \lim_{x\to 0^+}\frac{\ln(\frac{1}{x})}{\frac{1}{x}}$ $\quad \frac{\infty}{\infty}$

$\ln(\frac{1}{x}) = \overset{0}{\ln(1)} - \ln(x) = -\ln(x)$

$= \lim_{x\to 0^+}\frac{-\ln(x)}{x^{-1}}$

$\overset{L'H}{=} \lim_{x\to 0^+}\frac{-\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+}\frac{1}{x}\div\frac{1}{x^2} = \lim_{x\to 0^+} x = 0$.

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If the form is $\infty-\infty$ then we rewrite the expression as a single fraction, then apply L'H: ④

Ex) $\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$ $\quad \infty-\infty$

Find a common denominator:

$= \lim_{x\to 1^+}\left(\frac{\ln(x)}{\ln(x)}\cdot\frac{x}{(x-1)} - \left(\frac{x-1}{x-1}\right)\frac{1}{\ln(x)}\right)$

$= \lim_{x\to 1^+}\frac{x\cdot\ln(x) - x + 1}{\ln(x)(x-1)}$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 1^+}\frac{(1)\ln(x) + x\cdot\frac{1}{x} - 1 + 0}{\frac{1}{x}(x-1) + \ln(x)(1-0)}$

$= \lim_{x\to 1^+}\frac{\ln(x)+1-1}{1-\frac{1}{x}+\ln(x)} = \frac{\ln(1)}{1-1+\ln(1)} = \frac{0}{0}$

$= \lim_{x\to 1^+}\frac{\ln(x)}{1-x^{-1}+\ln(x)}$

$\overset{L'H}{=} \lim_{x\to 1^+}\frac{\frac{1}{x}}{0+x^{-2}+\frac{1}{x}} = \frac{1}{0+1+1} = \boxed{\frac{1}{2}}$

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If the form is 0^0 or ∞^0 or 1^∞, then we let y be the expression, take the natural logarithm of both sides & simplify:

Ex) $\lim_{x\to 0}(1+\tan x)^{\frac{1}{x}}$ $\quad 1^{\text{undefined}}$

let $y = (1+\tan x)^{\frac{1}{x}}$ $\quad$ • apply ln to both sides.
$\ln(y) = \ln(1+\tan x)^{\frac{1}{x}}$ $\quad$ • Property: $\ln a^b \to b\ln a$
$\ln(y) = \frac{1}{x}\ln(1+\tan x)$

So $\lim_{x\to 0}\ln(y) = \lim_{x\to 0}\frac{1}{x}\ln(1+\tan x)$ $\quad \frac{\ln(1)}{0} = \frac{0}{0}$

$= \lim_{x\to 0}\frac{\ln(1+\tan x)}{x}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{\frac{\sec^2 x}{1+\tan x}}{1}$

$= \frac{\sec^2 0}{1+\tan 0}$

$= \frac{1}{1+0}$

$= \boxed{1}$ $\quad \therefore \ln(y) = 1$, $\boxed{y = e^1}$ $\quad$ Apply e to get rid of ln

$\therefore \lim_{x\to 0}(1+\tan x)^{\frac{1}{x}} = e$.

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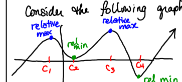
§4.4 Extreme Values

Consider the following graph of $y=f(x)$:

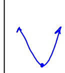


Def'n: • We say $x=c$ is a relative maximum if $f(c)\geq f(x)$ for all x-values near c.
• We say $x=c$ is a relative minimum if $f(c)\leq f(x)$ for all x-values near c.

How do we determine the relative min/max values of a function?
1) find $f'(x)=0$ & solve for x. We also find x-values that make $f'(x)$ undefined
2) We create a sign chart around these x-values.

Ex) Find the rel min/max of $y=x^2+2$
1) $f'(x) = 2x$
$2x = 0$
$x = 0$

2) Sign chart: $-\infty$ … 0 … ∞ (test value $x=-1$, test value $x=1$)



	$(-\infty,0)$	$(0,\infty)$
Sign of $f'(x)$	$f'(-1) = -2$ $-$	$f'(1) = 2$ $+$
Increasing or Decreasing	↘	↗

Since we go from decreasing to increasing, $x=0$ is a relative minimum.

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