$\mathbb{C}$, the Complex Numbers

Recall a basic (and I mean basic) polynomial $p(x) = x^2 - 1$. The roots, the x such that $p(x) = 0$, of $p(x)$ are ± 1. Recall that for $p(1) = 0$, $p(x)$ must contain at least one $(x - 1)$ such that

$$p(x) = (x - 1)\, p_1(x).$$

We only have the two roots, so we can factor to

$$p(x) = x^2 - 1 = (x - 1)(x + 1).$$

Next, let us consider $p(x) = x^2 + 1$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, for all x in the real numbers,

$$x^2 + 1 \geq 1 \quad \text{for all } x \in \mathbb{R}$$

so it has no real roots, none at all. What we need is some value, call it i, such that $i^2 = -1$, leading to

$$p(i) = (i)^2 + 1 = (-1) + 1 = 0.$$

This leads to the full factorization $p(x) = (x - i)(x + i)$.

Well, that problem is solved, in a sense, but i is NOT a real number, it is classified as 'imaginary'. In fact, it is frequently helpful to view all imaginary numbers as, in essence, at 'right angles to reality,' so in a totally separate direction than all real numbers. There is no reason we can't multiply i by real numbers, or why we can't add it to real numbers. Combining real and imaginary numbers creates a 'complex' number, which we will frequently call z. The basic form of is

$$z = a + bi,$$

a is the length in the real direction and b is the length in the imaginary direction. The typical visualization of complex numbers is on a Cartesian plane, with the horizontal axis representing the real numbers and the vertical axis the purely imaginary ones. Everything on this plane taken together forms the complex numbers, written $\mathbb{C}$.