Linear Independence

... and dependence, as well. They are quite heavily connected. Consider the following spanning set for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Do we really need all three of those terms to span the space? Any two of them will do (check, if you feel like it).

Look at the last two. They're in similar directions, but they are in DIFFERENT directions. As a result, a span of the two of them has two directions to work with and, as a result, covers a plane, in this case $\mathbb{R}^2$. The two vectors are in different directions because

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad a \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{for all } a \in \mathbb{R}.$$

Problem is, that's a property of just two vectors. It's not as much a property of the whole set, and there are two different equations to check. Notice, however, that there's one commonality between those two $\neq$ sets:

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{means that } a = b = 0$$

which is the exact same as

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{0} \qquad \text{means that } a = b = 0.$$

That statement is totally equivalent to the two earlier ones.

Now to bring back the full set of three:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{implies that} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{0}$$

so we have a linear combination of the three that adds to zero, but we didn't have to multiply them all by zero. The vectors going in different directions could only be linearly combined to $\mathbf{0}$ if they were all multiplied by zero. The set with an unnecessary term could add up to zero without resorting to multiplying by zero. It's not what you can make out of the vectors, it's how you can make it.

Official definition time:

Definition: a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is *Linearly Independent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_3 = \mathbf{0} \quad \text{implies that } a_1 = a_2 = \cdots = a_n = 0.$$

So, a linear combination of the vectors adding to zero has to be made using ONLY zero coefficients. Note that this linear combination is ALWAYS POSSIBLE. A set of vectors is linearly independent if that is the *only* option.