so we get the equations

$$a + 2c = 0 \qquad 2b + c = 0 \qquad -a + b - c = 0$$

so $c = -2b$ and $a = 4b$ which, in the last equation, produces $-4b + b + 2b = -b = 0$, so $a = b = c = 0$ and the set is linearly independent. In a three dimensional space any three linearly independent vectors form a basis, so done, they're a spanning set.

Dimension and Subspaces

Subspaces work exactly the way you would expect with bases and dimension.

Proposition: If U is a subspace of V then $\mathrm{Dim}(U) \leq \mathrm{Dim}(V)$.

Proof.
This one is actually fairly simple. Any basis of U is also a linearly independent set in V, and so must have fewer or the same number of elements as a basis of V. ■

Proposition: If U is a subspace of V and $\mathrm{Dim}(U) = \mathrm{Dim}(V)$ then $U = V$.

Proof.
Also fairly simple. If U has the same dimension as V then a basis of U must have as many elements as a basis of V, and be linearly independent. It becomes a maximal independent set in V and so is a basis. ■

The Main Point of Bases

Theorem
Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for vector space V. For any $\mathbf{x} \in V$ there are *unique* coefficients $a_1, \ldots, a_n$ such that

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Proof.
First, we can get some values a since the vectors span V. Now we need them to be unique. Assume we have another set of values:

$$\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$$

so

$$\mathbf{x} - \mathbf{x} = \mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$$

which is a linear combination for the zero vector, and the set of vectors is linearly independent. As a result, each coefficient $(a_k - b_k) = 0$, so the a and b are equal. The expansions are unique. ■