variable.

Variables that don't have a pivot are typically described as free variables, since the REF augmented matrix is basically written to make them the independent variables. Pivot variables are written as dependent variables.

Examples: The following matrices are in REF:

$$\begin{bmatrix} 1 & 2 & -4 & 12 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Lets write out the general solution of that last one.

First, we have x_4 as a non-pivot variable, so it's a free variable. Make it $x_4 = t$, so $x_3 + x_4 = 2$ so $x_3 = 2 - x_4 = 2 - t$.

Next, x_2 does not have a pivot, so make $x_2 = s$. Our final step is

$$x_1 = 1 + 3x_2 - 2x_3 \quad \Longrightarrow \quad x_1 = 1 + 3s - 2(2 - t) \quad \Longrightarrow \quad x_1 = -3 + 3s + 2t.$$

It would have been helpful if the variables with leading ones (the pivot variables) had no other pivot variables in their rows (i.e., they were all zero). This would involve having ONLY free variables accompanying the leading one in a row.

Definition: As augmented matrix is in *Reduced Row Echelon Form* if:

- It is in REF.
- Each pivot (leading one) has only zeros above and below (it's the only non-zero term on its column).

Here's one:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

So x_4 is free, make it $x_4 = t$ $x_3 = -t$, $x_2 = 1 - 2t$ and $x_1 = 4 + t$. We got those results earlier, but with more difficulty.