How to find the basis for Col(A): this is easy. We discussed this earlier. You row reduce A down to REF, and the pivot columns relate to the basis vectors. You have to use the ORIGINAL columns, but the row reduction decides which are necessary to the span (so, the minimal spanning set).

Theorem: The columns of A that hold pivots form a basis of Col(A).

Proof:
Fairly simple, and mostly involves reiterating stuff we've already seen. Take the linear system $[A|\mathbf{y}]$ and solve it. Removing the columns without pivots will not affect the span, since for any $\mathbf{y}$ where you have a solution you'd also have a solution with all the free variables set to zero. So, cut A back to B, which only has the pivot columns. Solving $[B|\mathbf{0}]$ will result only in pivot columns on the left, so the trivial solution, and the columns of B form a linearly independent spanning set and so a basis.

THE thing to remember is that the basis is the ORIGINAL column vectors, those in A as it started. The row reduction changes their shape.

Corollary: The dimension of Col(A) is equal to the number of pivots in A.

Example:

$$\text{Find a basis for } \operatorname{Col}(A) \text{ with } A = \begin{bmatrix} -1 & -1 & 0 & 2 & 2 \\ 2 & 1 & -1 & -1 & -1 \\ 1 & 0 & -1 & 2 & 3 \end{bmatrix}.$$

This one reduces to

$$\begin{bmatrix} 1 & 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \text{ so we have the basis } \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

And now for another space. This one is really only definable using the 'A as operator' setup.

Definition: The *Null Space* of A is the set

$$\operatorname{Null}(A) = \{\mathbf{x} \text{ such that } A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^m.$$

This is just a specific version of the Linear Independence equation for the columns of A. The order of the vectors is essential, however...

Finding the basis for this is not that easy, but it should be familiar. We just want the parametric vector form of the solution of $A\mathbf{x} = \mathbf{0}$. We've done this before.

Example: Find a basis for Null(A) using the A from before.