**Properties:**

- $A^{-1}$ exists implies $(A^T)^{-1} = (A^{-1})^T$.
- $A, B$, both invertible, implies $(AB)^{-1} = B^{-1}A^{-1}$.
- $A^{-1}$ exists means $(A^{-1})^{-1} = A$.
- $A$ invertible then $cA$ is too, for $c \in \mathbb{R} \neq 0$. $(cA)^{-1} = \frac{A^{-1}}{c}$.
- $I$ is invertible. $I^{-1} = I$.
- $A$ invertible then $A^k$ is invertible, $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ (use $A^0 = I$).
- If $A$ is invertible, its inverse is unique.

Now take a look at what this means for a linear system:

$$A\mathbf{x} = \mathbf{b}, \quad A \text{ invertible} \quad \Longrightarrow \quad A^{-1}Ax = A^{-1}b \quad \Longrightarrow \quad x = A^{-1}b.$$

Having an invertible $A$ on the left hand side means you can get the solution that way (usually not worth the effort, just solve it), so

- You ALWAYS get a solution (always consistent).
- The solution is unique.

This is primarily of value when one has to calculate many solutions of the form $A\mathbf{x} = \mathbf{b}$. Each can then be solved with a mere multiplication, nothing else.

**Calculating Inverses:** Not as hard as you may expect. First, it's really easy for any matrix smaller than $3 \times 3$.

For a $1 \times 1$, $[a]^{-1} = [\frac{1}{a}]$. Sorry, couldn't resist.

For a $2 \times 2$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

**Example:**

$$\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}^{-1} = \frac{1}{-6+4}\begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & -1 \end{bmatrix}.$$

Once you're dealing with a bigger matrix than that, you have to row reduce. For $A$, an $n \times n$ matrix, simply solve the linear system $[A|I]$, row reduce the left down to $I$ and the right will be $A^{-1}$.