Definition 4 A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive.

Example 2 Let T be the set of all triangles in a plane with R a relation in T given by R = {(T_1, T_2) : T_1 is congruent to T_2}. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 3 Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is perpendicular to L_2}. Show that R is symmetric but neither reflexive nor transitive.

Solution R is not reflexive, as a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. R is symmetric as $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

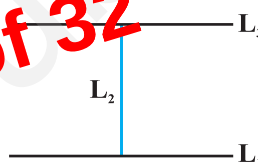

Fig 1.1

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

Example 4 Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.

Solution R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 5 Show that the relation R in the set **Z** of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Solution R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even (Why?). So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in **Z**.

1.4 Composition of Functions and Invertible Function

In this section, we will study composition of functions and the inverse of a bijective function. Consider the set A of all students, who appeared in Class X of a Board Examination in 2006. Each student appearing in the Board Examination is assigned a roll number by the Board which is written by the students in the answer script at the time of examination. In order to have confidentiality, the Board arranges to deface the roll numbers of students in the answer scripts and assigns a fake code number to each roll number. Let $B \subset \mathbf{N}$ be the set of all roll numbers and $C \subset \mathbf{N}$ be the set of all code numbers. This gives rise to two functions $f : A \to B$ and $g : B \to C$ given by $f(a)$ = the roll number assigned to the student a and $g(b)$ = the code number assigned to the roll number b. In this process each student is assigned a roll number through the function f and each roll number is assigned a code number through the function g. Thus, by the combination of these two functions, each student is eventually attached a code number.

This leads to the following definition:

Definition 8 Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)), \forall\, x \in A.$$

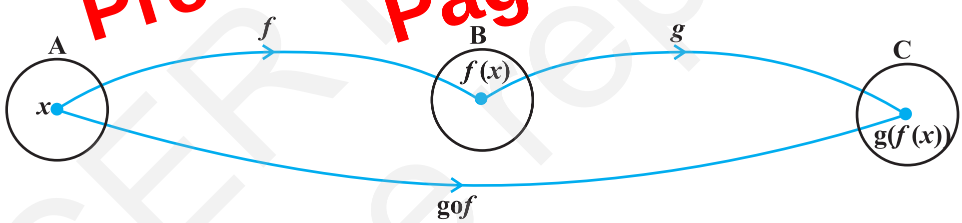

Fig 1.5

Example 15 Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Definition 9 A function $f : \mathrm{X} \rightarrow \mathrm{Y}$ is defined to be *invertible*, if there exists a function $g : \mathrm{Y} \rightarrow \mathrm{X}$ such that $gof = \mathrm{I_X}$ and $fog = \mathrm{I_Y}$. The function g is called the *inverse of f* and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 23 Let $f : \mathbf{N} \rightarrow \mathrm{Y}$ be a function defined as $f(x) = 4x + 3$, where, $\mathrm{Y} = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$ for some x in the domain $\mathbf{N}$. This shows that $x = \frac{(y-3)}{4}$. Define $g : \mathrm{Y} \rightarrow \mathbf{N}$ by $g(y) = \frac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$ and $fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y$. This shows that $gof = \mathrm{I_N}$ and $fog = \mathrm{I_Y}$, which implies that f is invertible and g is the inverse of f.

Example 24 Let $\mathrm{Y} = \{n^2 : n \in \mathbf{N}\} \subset \mathbf{N}$. Consider $f : \mathbf{N} \rightarrow \mathrm{Y}$ as $f(n) = n^2$. Show that f is invertible. Find the inverse of f.

Solution An arbitrary element y in Y is of the form n^2, for some $n \in \mathbf{N}$. This implies that $n = \sqrt{y}$. This gives a function $g : \mathrm{Y} \rightarrow \mathbf{N}$, defined by $g(y) = \sqrt{y}$. Now, $gof(n) = g(n^2) = \sqrt{n^2} = n$ and $fog(y) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$, which shows that $gof = \mathrm{I_N}$ and $fog = \mathrm{I_Y}$. Hence, f is invertible with $f^{-1} = g$.

Example 25 Let $f : \mathbf{N} \rightarrow \mathbf{R}$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : \mathbf{N} \rightarrow \mathrm{S}$, where, S is the range of f, is invertible. Find the inverse of f.

Solution Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in $\mathbf{N}$, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{\left(\sqrt{y-6}\right) - 3}{2}$, as $y \geq 6$.

Let us define $g : \mathrm{S} \to \mathbf{N}$ by $g(y) = \dfrac{\sqrt{y-6}-3}{2}$.

Now $\quad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$$= \frac{\left(\left(\sqrt{(2x+3)^2+6-6}\right)-3\right)}{2} = \frac{(2x+3-3)}{2} = x$$

and $\quad fog(y) = f\left(\dfrac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}\right) = \left(\dfrac{2\left(\left(\sqrt{y-6}\right)-3\right)}{2}+3\right)^2 + 6$

$$= \left(\left(\sqrt{y-6}\right)-3+3\right)^2 + 6 = \left(\sqrt{y-6}\right)^2 + 6 = y - 6 + 6 = y.$$

Hence, $\quad gof = \mathrm{I_N}$ and $fog = \mathrm{I_S}$. This implies that f is invertible with $f^{-1} = g$.

Example 26 Consider $f : \mathbf{N} \to \mathbf{N}$, $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{R}$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall$ x, y and z in N. Show that $ho(gof) = (hog)of$.

Solution We have

$$ho(gof)(x) = h(gof(x)) = h(g(f(x))) = h(g(2x))$$
$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4) \quad \forall\, x \in \mathbf{N}.$$

Also, $\quad ((hog)\,o\,f)(x) = (hog)(f(x)) = (hog)(2x) = h(g(2x))$

$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4), \ \forall\, x \in \mathbf{N}.$$

This shows that $ho(gof) = (hog)\,o\,f$.

This result is true in general situation as well.

Theorem 1 If $f : \mathrm{X} \to \mathrm{Y}$, $g : \mathrm{Y} \to \mathrm{Z}$ and $h : \mathrm{Z} \to \mathrm{S}$ are functions, then

$$ho(gof) = (hog)\,o\,f.$$

Proof We have

$$ho(gof)(x) = h(gof(x)) = h(g(f(x))), \ \forall\, x \text{ in X}$$

and $\quad (hog)\,of\,(x) = hog(f(x)) = h(g(f(x))), \ \forall\, x$ in X.

Hence, $\quad ho(gof) = (hog)\,o\,f$.

Example 27 Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to$ {apple, ball, cat} defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a)$ = apple, $g(b)$ = ball and $g(c)$ = cat. Show that f, g and gof are invertible. Find out f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1} o\, g^{-1}$.

18. Let $f : \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : \mathrm{R} \to \mathrm{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does *fog* and *gof* coincide in $(0, 1]$?

19. Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by $\mathrm{R} = \phi \subset \mathrm{X} \times \mathrm{X}$.
- *Universal relation* is the relation R in X given by $\mathrm{R} = \mathrm{X} \times \mathrm{X}$.
- *Reflexive relation* R in X is a relation with $(a, a) \in \mathrm{R}\ \forall\, a \in \mathrm{X}$.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in \mathrm{R}$ implies $(b, a) \in \mathrm{R}$.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in \mathrm{R}$ and $(b, c) \in \mathrm{R}$ implies that $(a, c) \in \mathrm{R}$.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in \mathrm{X}$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *one-one* (or *injective*) if
  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in \mathrm{X}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *onto* (or *surjective*) if given any $y \in \mathrm{Y}, \exists\, x \in \mathrm{X}$ such that $f(x) = y$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- The *composition* of functions $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ is the function $gof : \mathrm{A} \to \mathrm{C}$ given by $gof(x) = g(f(x))\ \forall\ x \in \mathrm{A}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *invertible* if $\exists\, g : \mathrm{Y} \to \mathrm{X}$ such that $gof = \mathrm{I_X}$ and $fog = \mathrm{I_Y}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *invertible* if and only if f is one-one and onto.