Solution The information is represented in the form of a 3 × 2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

Example 3 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\qquad a_{ij} = \frac{1}{2}|i - 3j|$, i = 1, 2, 3 and j = 1, 2.

Therefore $\qquad a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \qquad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \qquad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $\quad (X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or $\quad X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $\quad (X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $\quad (X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$

or $\quad Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$. Find the total amount spent by the group in the two cities X and Y.

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A+B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute 3A – 5B.

6. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

7. Find X and Y, if

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

8. Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

9. Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

12. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w.

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs 80, Rs 60 and Rs 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix 7X – 5Z is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A′ or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 *Properties of transpose of the matrices*

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A\,B)' = B'\,A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as $B' = -B$

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

Theorem 1 For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof Let $B = A + A'$, then

$$\begin{aligned} B' &= (A + A')' \\ &= A' + (A')' \quad (\text{as } (A + B)' = A' + B') \\ &= A' + A \quad (\text{as } (A')' = A) \\ &= A + A' \quad (\text{as } A + B = B + A) \\ &= B \end{aligned}$$

Therefore $\quad B = A + A'$ is a symmetric matrix

Now let $\quad C = A - A'$

$$\begin{aligned} C' &= (A - A')' = A' - (A')' \quad (\text{Why?}) \\ &= A' - A \quad (\text{Why?}) \\ &= -(A - A') = -C \end{aligned}$$

Therefore $\quad C = A - A'$ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let A be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that $(A + A')$ is a symmetric matrix and $(A - A')$ is a skew symmetric matrix. Since for any matrix A, $(kA)' = kA'$, it follows that $\frac{1}{2}(A + A')$ is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Miscellaneous Examples

Example 26 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, $\quad$ the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k+1$

Now $\quad A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\,\theta & \sin n\,\theta \\ -\sin n\,\theta & \cos n\,\theta \end{bmatrix}$, holds for all natural numbers.

Example 27 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore A′ = A and B′ = B.

Let $\quad$ AB be symmetric, then (AB)′ = AB